

MATH 125

Solutions to Midterm 1 (Drogon)

1. A tank holding 1200 gallons of maple syrup can be drained completely in three hours by opening a valve at its bottom. The amount of syrup in the tank at time t (where t is in hours, with $0 \leq t \leq 3$) is given by

$$V(t) = 1200 \left(1 - \frac{t}{3}\right)^2 = \frac{1200}{9}(3-t)^2.$$

5 points

- (a) Calculate the average rate (in gallons per hour) at which the syrup drains between $t = 1$ hour and $t = 2$ hours.

Solution: This is just the change in volume divided by the change in time. Since the change in time is 1 hour, we have

$$\text{average rate} = \frac{V(2) - V(1)}{2 - 1} = \frac{\frac{1200}{9} - \frac{1200}{9} \cdot 4}{1} = -1200/3 = -400 \text{ gal/hr.}$$

(A positive answer is also correct, depending on how you interpret the phrase “rate at which it is draining”.)

5 points

- (b) Write a formula that represents the average rate (in gallons per hour) at which the syrup drains between 2 hours some time t hours (with $0 \leq t \leq 3, t \neq 2$).

Solution: As above, we have

$$\begin{aligned} \text{average rate} &= \frac{V(t) - V(2)}{t - 2} = \frac{1200}{9} \left(\frac{(3-t)^2 - 1}{t - 2} \right) = \frac{1200}{9} \left(\frac{9 - 6t + t^2 - 1}{t - 2} \right) \\ &= \frac{1200}{9} \left(\frac{8 - 6t + t^2}{t - 2} \right) \\ &= \frac{1200}{9} \left(\frac{(t-2)(t-4)}{t-2} \right) \\ &= \frac{1200}{9}(t-4). \end{aligned}$$

(Depending on how you interpret it, $\frac{1200}{9}(4-t)$ is also correct.)

5 points

- (c) Calculate the instantaneous rate (in gallons per hour) at which the syrup is draining out at time $t = 2$ hours (depending on how you write it, this could be negative).

Solution: This is just the limit of the previous part as $x \rightarrow 2$. We have

$$\lim_{x \rightarrow 2} \frac{1200}{9}(x-4) = \frac{1200}{9}(-2) = -\frac{800}{3} \frac{\text{gal}}{\text{hr}}$$

2. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

5 points

(a) $\lim_{x \rightarrow \pi/4} x^2 \tan x$

Solution: Since the function is continuous at $\pi/4$, we just evaluate:

$$\left(\frac{\pi}{4}\right)^2 \tan\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16} \cdot 1 = \frac{\pi^2}{16}.$$

5 points

(b) $\lim_{x \rightarrow 4} \frac{x^2 - 16}{5x^2 - 20x}$

Solution:

$$\lim_{x \rightarrow 4} \frac{(x-4)(x+4)}{5x(x-4)} = \lim_{x \rightarrow 4} \frac{(x+4)}{5x} = \frac{4+4}{20} = \frac{2}{5}.$$

5 points

(c) $\lim_{x \rightarrow 3^+} \frac{x^3 - 4x^2 + 1}{(x-5)(x-3)}$

Solution: Note for x close to 3, the numerator is close to -8 while the denominator tends towards zero. Thus, the function becomes unbounded at 3. Since we are looking at values of $x > 3$, the denominator is always negative. Hence, the limit is $+\infty$.

3. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

5 points

(a) $\lim_{x \rightarrow -\infty} \frac{9x^5 - 1}{3x^5 - 27x^3 + 10}$

Solution: Since x is tending to infinity, the x^5 term dominates the others, so we have

$$\lim_{x \rightarrow -\infty} \frac{9x^5 - 1}{3x^5 - 27x^3 + 10} = \lim_{x \rightarrow -\infty} \frac{9x^5}{3x^5} = \lim_{x \rightarrow -\infty} \frac{9}{3} = 3.$$

5 points

(b) $\lim_{x \rightarrow 4} \frac{x^2 - 16}{|x - 4|}$

Solution: Because of the absolute value, we need to consider $x < 4$ and $x > 4$ separately. For $x > 4$, $|x - 4| = (x - 4)$, and we get

$$\lim_{x \rightarrow 4^+} \frac{x - 16}{|x - 4|} = \lim_{x \rightarrow 4^+} \frac{(x - 4)(x + 4)}{x - 4} = \lim_{x \rightarrow 4^+} x + 4 = 8.$$

When $x < 4$, $|x - 4| = -(x - 4)$, yielding

$$\lim_{x \rightarrow 4^-} \frac{x - 16}{|x - 4|} = \lim_{x \rightarrow 4^-} -(x + 4) = -8.$$

Since the two one-sided limits differ, the limit does not exist.

5 points

(c) $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$

Solution: We need to use the conjugate to deal with the square root:

$$\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} \left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right) = \lim_{x \rightarrow 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{6}.$$

4. For $0 < x < 2$, let $h(x) = \frac{4}{x} + \frac{4}{x - 2} + \cos(\pi x)$.

2 points

- (a) What is
- $h(1)$
- ? Simplify as much as possible, or write "Not Defined" if it is not defined.

Solution: $h(1) = 4 - 4 + \cos(\pi) = -1$.

5 points

- (b) Calculate
- $\lim_{x \rightarrow 0^+} h(x)$
- and
- $\lim_{x \rightarrow 2^-} h(x)$
- . If either limit is not a number, distinguish between
- $+\infty$
- ,
- $-\infty$
- , and DNE (does not exist).

Solution: $\lim_{x \rightarrow 0^+} h(x) = +\infty$ (because for x small but positive, $1/x$ is extremely large and positive while $-\frac{1}{x-2}$ is close to $-1/2$ and $\cos(\pi x)$ is close to 1).

Similarly, $\lim_{x \rightarrow 2^-} h(x) = -\infty$.

7 points

- (c) Is there a value of
- x
- between 0 and 2 so that
- $h(x) = 0$
- ?

If so, is the value of x bigger than, equal to, or less than 1? If not, why not?

How do you know? (Fully justify your answer.)

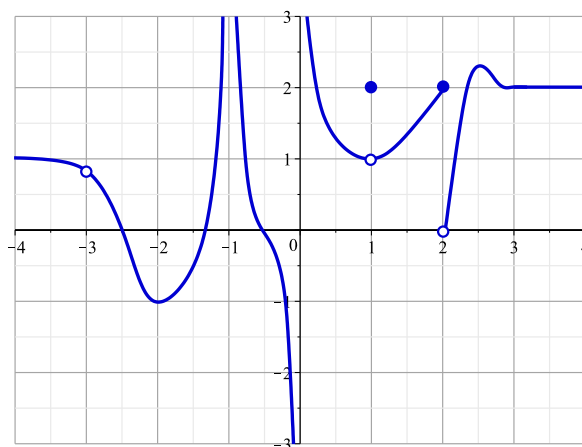
Solution: Since $h(1) < 0$ and $\lim_{x \rightarrow 0^+} h(x) = +\infty$, we know there are values of h that are positive for x close to 0. Since $h(1) < 0$ and h is continuous, there must be an x between 0 and 1 with $h(x) = 0$ by the Intermediate Value Theorem.

10 points 5. Explain in words what $\lim_{x \rightarrow 3} f(x) = +\infty$ means.

Solution: This just means that $f(x)$ becomes arbitrarily large for values of x sufficiently close to 3.

Alternatively, we could say that f has a vertical asymptote at $x = 3$, but then we would also need to specify that f increases for $x < 3$ and decreases for $x > 3$ as well. This is necessary in order to distinguish between cases like $\lim_{x \rightarrow 3} f(x) = -\infty$, or when we have $\lim_{x \rightarrow 3^+} f(x) = +\infty$ but $\lim_{x \rightarrow 3^-} f(x) = -\infty$, etc., which also have a vertical asymptote at $x = 3$, but $\lim_{x \rightarrow 3} f(x) \neq +\infty$.

20 points 6. The graph of a function $f(x)$ is shown at right, for $-4 < x < 4$. Answer the questions below, assuming that for values not shown on the graph, the function continues in the same way (i.e., essentially straight). When calculating limits, distinguish between $+\infty$, $-\infty$, and “does not exist”.



(a) What is $\lim_{x \rightarrow +\infty} f(x)$, if it exists?

Solution: f has a horizontal asymptote on the right at $y = 2$, so $\lim_{x \rightarrow +\infty} f(x) = 2$.

(b) What is $\lim_{x \rightarrow 2^+} f(x)$, if it exists?

Solution: As x tends to 2 from above, the graph limits on $y = 0$, so $\lim_{x \rightarrow 2^+} f(x) = 0$.

(c) What is $\lim_{x \rightarrow 2} [(x-2)f(x)]$, if it exists?

Solution: Note that for $1 \leq x \leq 3$, we have $0 \leq f(x) \leq 3$. This means that we also have $0 \leq (x-2)f(x) \leq 3(x-2)$. Using the Squeeze Theorem, we have

$$0 \leq \lim_{x \rightarrow 2} [(x-2)f(x)] \leq \lim_{x \rightarrow 2} 3(x-2) = 0,$$

so the limit is 0.

(d) List all points x between -4 and 4 where $f(x)$ is defined, but $f(x)$ is not continuous.

Solution: $f(x)$ is not continuous at $x = -3$, $x = -1$, $x = 0$, $x = 1$ and $x = 2$. However, it is also not defined when x is one of the points $-3, -1$, or 0 , so the answer is only $x = 1$ and $x = 2$.

- 10 points 7. What value of k is necessary so that the function

$$f(x) = \begin{cases} kx + 5 + \ln 2 & x < 1 \\ 2kx^2 + x & x \geq 1 \end{cases}$$

is continuous for all positive values of x ? If there is no such value, write "NONE". Justify your answer fully.

Solution: The function f is continuous at all positive values of x except possibly at $x = 1$, since $kx + 5 + \ln(2)$ and $2kx^2 + x$ continuous all values of x . We just need to choose k so that the ends meet.

Since $f(1) = 2k + 1$, we need to find k so that $\lim_{x \rightarrow 1^-} f(x) = 2k + 1$.

Observe that

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} kx + 5 + \ln 2 = k + 5 + \ln 2,$$

so we need to take k so that $2k + 1 = k + 5 + \ln 2$, that is, $k = 5 - 1 + \ln(2) = 4 + \ln 2$.

- 10 points 8. Write a limit that represents the slope of the line tangent to the graph of the function

$$f(x) = \begin{cases} (1-x) \tan(\frac{\pi}{2}x) & x \neq 1 \\ \frac{2}{\pi} & x = 1 \end{cases}$$

at $x = 1$. You **do not need to evaluate the limit**.

Solution: We just use the definition of the derivative at $x = 1$:

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}.$$

Since h is not zero, $f(1+h) = h \tan(\frac{\pi}{2}(1+h))$ and $f(1) = 2/\pi$. So,

$$f'(1) = \lim_{h \rightarrow 0} \frac{h \tan(\frac{\pi}{2}(1+h)) - 2/\pi}{h}.$$

If you prefer to use the version of the definition with $x \rightarrow 1$, you would have

$$f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{(1-x) \tan(\pi x/2) - 2/\pi}{x - 1}.$$