Math 125 - Fall 2006
Solutions to the Practice Final Examination

1. Let f be a continuous function. Find

$$\lim_{x\to\infty} f\left((1-\tfrac{1}{x})^x\right).$$

Solution: Since f is continuous, we have

$$\lim_{x\to\infty} f\left((1-\tfrac{1}{x})^x\right) = f\left(\lim_{x\to\infty}(1-\tfrac{1}{x})^x\right).$$

Now,

$$\ln(1-1/x)^x = x\ln(1-1/x) = \frac{\ln[(x-1)/(x)]}{1/x} = \frac{\ln(x-1)-\ln(x)}{1/x}.$$

Using L'Hospital's rule, we have

$$\lim_{x\to\infty}\ln(1-1/x)^x = \lim_{x\to\infty}\frac{\frac{1}{x-1}-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -x^2\frac{x-(x-1)}{x(x-1)} = \lim_{x\to\infty} -\frac{x}{x-1} = -1.$$

Thus $\lim_{x\to\infty}(1-\frac{1}{x})^x = e^{-1}$, and so the answer is $f(e^{-1})$.

2. Consider the equation $x + e^x = 0$. Is there a solution to this equation? Why or why not.

Solution: Let $f(x) = x + e^x$. Then $f(-1) = -1 + 1/e < 0$ while $f(0) = 1 > 0$. By the Intermediate Value Theorem, $f(x) = 0$ has a solution between 0 and -1.

3. Find the derivative of the function

$$e^{2\tan(\sqrt{x})}.$$

Solution:

$$\frac{d}{dx}e^{2\tan(\sqrt{x})} = e^{2\tan(\sqrt{x})}2(\sec(\sqrt{x}))^2 \times \left(\frac{1}{2\sqrt{x}}\right).$$

4. Consider the function

$$f(x) = \begin{cases} \frac{\sin x}{x} & x < 0 \\ x^3 + 2x + 1 & x \geq 0 \end{cases}$$

At which points is f continuous? At which points is it differentiable?

Solution: The only point where we have to worry is $x = 0$. Note that

$$\lim_{x\to 0^-} f(x) = 1 \quad \text{while} \quad \lim_{x\to 0^+} f(x) = 1.$$

Thus f is continuous at zero, and hence it is continuous everywhere.

Next,

$$f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & x < 0 \\ \\ 3x^2 + 2 & x \geq 0 \end{cases}$$

Thus

$$\lim_{x \to 0^+} f'(x) = 2$$

while

$$\begin{aligned} \lim_{x \to 0^-} f'(x) &= \lim_{x \to 0^-} x\sin x + \cos x - \cos x 2x \\ &= \lim_{x \to 0^-} x\sin x + \cos x - \cos x 2x \\ &= \lim_{x \to 0^-} x \sin x 2x \\ &= \lim_{x \to 0^-} \sin x 2 = 0. \end{aligned}$$

Thus f is not differentiable at $x = 0$.

5. Let $f(x) = x \ln\left(1 + e^{x^2}\right)$. Find $f'(5)$.

   Solution:

$$f'(x) = \ln\left(1 + e^{x^2}\right) + \frac{x}{1 + e^{x^2}} \cdot e^{x^2} \cdot 2x.$$

   Thus $f'(5) = \ln(1 + e^{25}) + \frac{5}{1+e^{25}} \cdot e^{25} \cdot 10$.

6. Show that the curves

$$e^{x^2 - y^2} \cos(2xy) = 1 \quad \text{and} \quad e^{x^2 - y^2} \sin(2xy) = 0$$

   meet orthogonally at the point $(\sqrt{\pi}, \sqrt{\pi})$.

   Solution: Let y_1 and y_2 be the functions determined by the first and second equations respectively. By implicit differentiation, we have

$$e^{x^2 - y_1^2} \cos(2xy_1)(2x - 2y_1 y_1') - e^{x^2 - y_1^2} \sin(2xy_1)(2y_1 + 2xy_1') = 0.$$

   Thus at $x = \sqrt{\pi}$ and $y_1 = \sqrt{\pi}$ we have

$$0 = e^{\pi - \pi} \cos(2\pi)(2\sqrt{\pi} - 2\sqrt{\pi} y_1') - e^{\pi - \pi} \sin(2\pi)(2\sqrt{\pi} + 2\sqrt{\pi} y_1') = 2\sqrt{\pi}(1 - y_1').$$

   Thus $y_1' = 1$.

   Similarly

$$e^{x^2 - y_2^2} \sin(2xy_2)(2x - 2y_2 y_2') + e^{x^2 - y_2^2} \cos(2xy_2)(2y_2 + 2xy_2') = 0.$$

Thus at $x = \sqrt{\pi}$ and $y_1 = \sqrt{\pi}$ we have

$$0 = e^{\pi-\pi}\sin(2\pi)(2\sqrt{\pi} - 2\sqrt{\pi}y_2') - e^{\pi-\pi}\cos(2\pi)(2\sqrt{\pi} + 2\sqrt{\pi}y_2') = 2\sqrt{\pi}(1 + y_2').$$

Thus $y_2' = -1$. This means the slopes of the two tangent lines to the respective curves are reciprocal, so that the lines are indeed orthogonal, as desired.

7. Find the derivative of the function

$$f(x) = \frac{(\sin x)^2(\tan x)^2}{(x^2+1)^2}.$$

Solution: take $\ln$ of both sides to get

$$\ln f(x) = 2\ln\sin x + 2\ln\tan x - 2\ln(x^2+1).$$

Then

$$\frac{f'(x)}{f(x)} = 2\frac{\cos(x)}{\sin x} + 2\frac{(\sec(x))^2}{\tan x} - \frac{4x}{x^2+1}.$$

Thus

$$f'(x) = \left(2\frac{\cos(x)}{\sin x} + 2\frac{(\sec(x))^2}{\tan x} - \frac{4x}{x^2+1}\right)\left(\frac{(\sin x)^2(\tan x)^2}{(x^2+1)^2}\right).$$

8. Find an equation for the tangent line to the curve

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

through the point $(0, 0.5)$.

Solution: By implicit differentiation, we have

$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x - 4yy' - 1),$$

so at $x = 0, y = 0.5$ we have

$$y' = 2(0.5)(2y' - 1).$$

Solving, we get $y' = 1$, so the equation for the tangent line is

$$\frac{Y - 0.5}{X} = 1.$$

9. If $f(x) = e^x/(x+1)^3$, find $f'(x)$ and $f''(x)$.

Solution:

using the quotient rule, we get

$$f'(x) = \frac{e^x(x+1)^3 - 3e^x(x+1)^2}{(x+1)^6} = \frac{e^x(x+1) - 3e^x}{(x+1)^4} = \frac{e^x(x-2)}{(x+1)^4}$$

$$f''(x) = \frac{(e^x(x-2) + e^x)(x+1)^4 - 4e^x(x-2)(x+1)^3}{(x+1)^8} = \frac{(xe^x - 2e^x)(x+1) - 4e^x(x-2)}{(x+1)^5} = \frac{e^x(x-2)(x-3)}{(x+1)^5}$$

10. Find the limit
$$\lim_{x\to 1} \frac{x^\pi - 1}{x^e - 1}.$$

Solution: This is the indeterminate form $0/0$. By L'Hospital's rule we have
$$\lim_{x\to 1} \frac{x^\pi - 1}{x^e - 1} = \lim_{x\to 1} \frac{\pi x^{\pi-1}}{e x^{e-1}} = \frac{\pi}{e}.$$

11. Show that $e^x \geq 1 + x$ for $x \geq 0$. (Hint: Consider the function $f(x) = e^x - 1 - x$.)

Solution: We are trying to show that $f(x) \geq 0$ for $x \geq 0$. Now, $f(0) = 1 - 1 - 0 = 0$, and $f'(x) = e^x - 1 \geq 0$ for $x \geq 0$. By the mean value theorem, there is some $c \in [0, x]$ so that
$$\frac{f(x) - f(0)}{x - 0} = f'(c) \geq 0.$$

It follows that $f(x) \geq f(0) = 0$ for $x \geq 0$, as desired.

12. A particle is moving along the curve $y = x^2$. As it passes through the point $(2, 4)$, its y coordinate changes at a rate of 5 m/sec. What is the rate of change of the particle's distance to the origin at this instant?

Solution: The distance D to the origin is
$$D = \sqrt{x^2 + y^2} = \sqrt{x^2 + x^4}.$$

Thus
$$\frac{dD}{dt} = \frac{2x + 4x^3}{2\sqrt{x^2 + x^4}} \frac{dx}{dt}.$$

Now $\frac{dy}{dt} = 2x\frac{dx}{dt}$. Thus
$$\frac{dD}{dt} = \frac{1 + 2x^2}{2\sqrt{x^2 + x^4}} 2x \frac{dx}{dt} = \frac{1 + 2x^2}{2\sqrt{x^2 + x^4}} \frac{dy}{dt}.$$

At $x = 2$ and $y = 4$ we have
$$\frac{dD}{dt} = \frac{9}{2\sqrt{20}} 5 = \frac{45}{4\sqrt{5}} m/sec.$$

13. Find the absolute maximum and absolute minimum values of the function
$$f(x) = x^2 - \ln x^2$$

on the interval $[1/4, 4]$.

Solution: $f'(x) = 2x - \frac{2}{x} = 0$ at $x = \pm 1$. Since we are in the domain $[1/4, 4]$, the only critical point is $x = 1$. Now,

$$f(1) = 1, \quad f(1/4) = \frac{1}{16} + \ln(16) \quad \text{and} \quad f(4) = 16 - \ln(16).$$

Thus the absolute max of f occurs at 4 and the absolute min occurs at 1. So to answer the question,

$$f_{max} = 16 - \ln 16 \quad \text{and} \quad f_{min} = 1.$$

14. Find

$$\lim_{x \to \frac{\pi}{2}} \tan(7x)\cos(4x).$$

Solution: This question was mistyped, but as is the solution is that the limit does not exist, since $\lim_{x \to \frac{\pi}{2}^{\pm}} \tan(7x) = \pm\infty$ and $\lim_{x \to \frac{\pi}{2}} \cos(4x) = 1$. The question should have been : Find

$$\lim_{x \to \frac{\pi}{2}} \tan(7x)\sin(4x).$$

In this case, we use L'Hospital's rule:

$$\begin{aligned}
\lim_{x \to \frac{\pi}{2}} \tan(7x)\sin(4x) &= \lim_{x \to \frac{\pi}{2}} \frac{\sin(7x)\sin(4x)}{\cos(7x)} \\
&= \lim_{x \to \frac{\pi}{2}} \frac{\sin(7x)\sin(4x)}{\cos(7x)} \\
&= \lim_{x \to \frac{\pi}{2}} \frac{7\cos(7x)\sin(4x) + 4\sin(7x)\cos(4x)}{-7\sin(7x)} = -\frac{4}{7}.
\end{aligned}$$

15. A woman wants to get from a point A on the shore of a circular lake to a point C diametrically opposite A in the shortest possible time. She can walk at a speed of $4\ mi/hr$ and row at a speed of $2\ mi/hr$. How should she proceed?

Solution: Let R be the radius of the lake. Suppose the woman goes an angle θ around the lake by foot, and then rows the rest of the time. Then the time it takes her to cross the lake is

$$T(\theta) = 4 \times R\theta + 4\sqrt{2}R \times \sqrt{1 + \cos\theta}.$$

(One has to use the cosine law to determine the length of her rowing route. It is the side of an isoceles triangle with two sides R and the angle between these two sides equal to $\pi - \theta$. Draw the picture and you will see this.) Thus

$$T'(\theta) = 4R\left(1 - \frac{\sqrt{2}\sin\theta}{2\sqrt{1 + \cos\theta}}\right).$$

Setting $T'(\theta) = 0$, we find

$$\sin\theta = \sqrt{2(1+\cos\theta)}.$$

Thus

$$(1+\cos\theta)(1-\cos\theta) = (\sin\theta)^2 = 2(1+\cos\theta).$$

Thus either $\cos\theta = -1$. It follows that the shortest time is to just walk and never row. This is also obvious if you think about it without using calculus.

16. Consider the function

$$f(x) = x^3 - 7x^2 + 9x - \pi.$$

(i) Find all the critical points of f, and the values of f at those points. State weather these points are local maxima, local minima or neither.

(ii) Find all the inflection of points of f.

Solution:

(i): $f'(x) = 3x^2 - 14x + 9 = 0$ precisely if

$$x = \frac{14 \pm \sqrt{88}}{6} = \frac{7 \pm \sqrt{22}}{3}.$$

Now, $f''(x) = 6x - 14$, which is positive at the positive root of $f'(x)$ and negative at the negative root. It follows that $\frac{7+\sqrt{22}}{3}$ is a local min and $\frac{7-\sqrt{22}}{3}$ is a local max. I leave it to the reader to calculate $f(\frac{7+\sqrt{22}}{3})$ and $f(\frac{7+\sqrt{22}}{3})$. (Note: Since

$$\lim_{x\to\pm\infty} f(x) = \pm\infty,$$

f has no absolute max or min.)

(ii): An inflection point occurs at the zero of f'', which in this case is the point

$$x = 7/3.$$