MAT125 Spring 2014 Final

Your name: SOLUTIONS

TA's name:

SPRING 2014

Problem #1: Use the definition of the derivative to find $f'(x)$

if $f(x) = 3x^2 - 5x + 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 5(x+h) + 1 - (3x^2 - 5x + 1)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 \boxed{- 5x} - 5h + 1 - 3x^2 \boxed{+ 5x} - 1}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 5h}{h} = \lim_{h \to 0} (6x + 3h - 5)$$

$$= 6x - 5$$

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Problem #2: Find $\frac{dy}{dx}$.

a) $y = \frac{x^2-3x+5}{\ln x}$

$$\frac{(2x-3)(\ln x) - (x^2-3x+5)\left(\frac{1}{x}\right)}{(\ln x)^2}$$

b) $y = e^{\tan 2x}$

$$\sec^2(2x) \cdot e^{\tan 2x}$$

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c) $y = \sqrt{\frac{2x+3}{2x-3}} = \left(\frac{2x+3}{2x-3}\right)^{1/2}$

$$y' = \frac{1}{2}\left(\frac{2x+3}{2x-3}\right)^{-1/2}\left(\frac{2(2x-3) - 2(2x+3)}{(2x-3)^2}\right)$$

$$= \frac{-6}{(2x-3)^2\sqrt{\frac{2x+3}{2x-3}}}$$

d) $y = \tan^{-1}(\pi x)$

$$\frac{\pi}{1 + (\pi x)^2}$$

Problem #3: Find the equation of the tangent line to $x^2 - 2xy + y^2 = 0$ at the point (1,1).

BY IMPLICIT DIFF

$$= 2x - (2y' + 2y) + 2yy'$$

AT (1,1), WE HAVE

MINUS

$$0 = 2 - 2y' \cancel{+} 2 \quad + 2y'$$

$$0 = 0$$

OH DEAR! LET'S THINK A BIT.

$x^2 - 2xy + y^2 = (x-y)^2$, SO IF $(x-y)^2 = 0$, THEN

$x - y = 0$ OR $-x + y = 0$

IE $y = x$

SINCE THIS IMPLICIT EQUATION DESCRIBES THE LINE

$$y = x$$

THE TANGENT @ (1,1) IS THE LINE

$$y = x.$$

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Problem #4: Graph $y = x^3 + 3x^2 - 24x + 12$. Be sure to label all extrema and points of inflection. You do not need to graph the x-intercepts.

$f(x) = x^3 + 3x^2 - 24x + 12$

$f'(x) = 3x^2 + 6x - 24 = 3(x+4)(x-2)$

$\Rightarrow$ CRITICAL POINTS AT $x = -4$, $x = 2$.

$f''(x) = 6x + 6.$

$\Rightarrow$ $\boxed{\text{INFLECTION POINT AT } x = -1}$

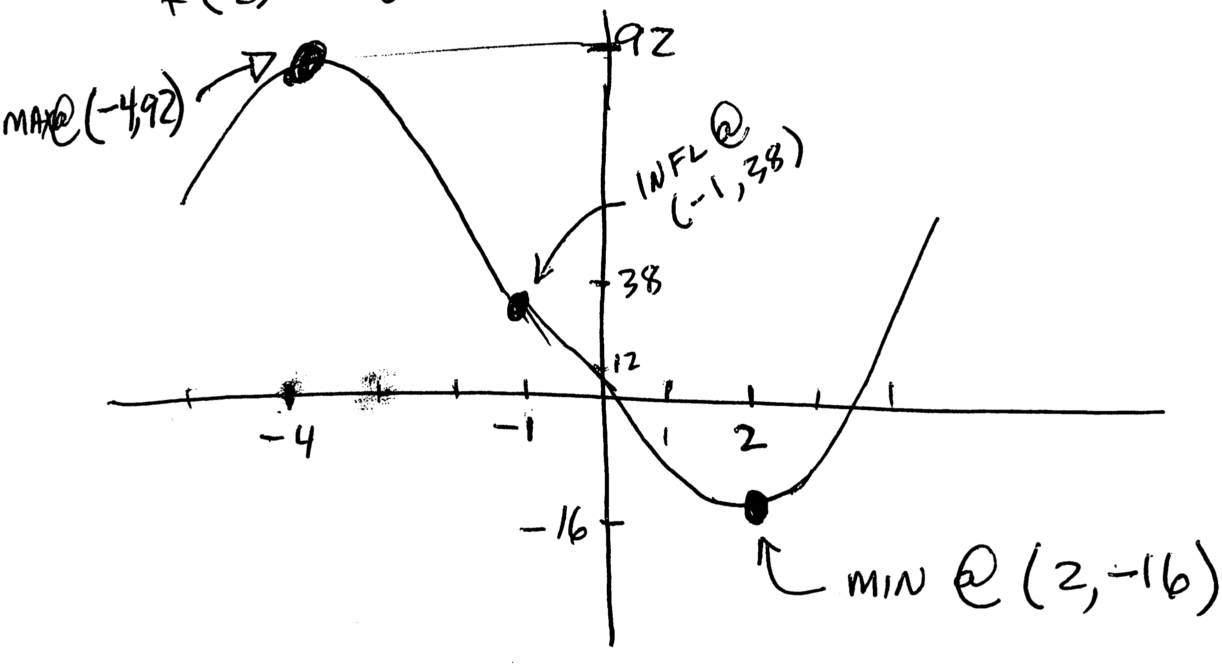
$f''(-4) = -24 + 6 < 0$, SO $x = -4$ IS A REL MAX

$f''(2) = 12 + 6 > 0$, SO $x = 2$ IS A REL MIN.

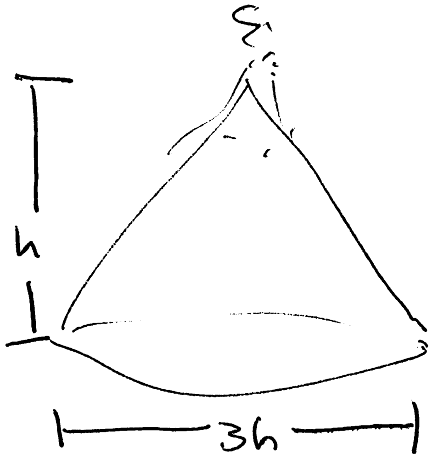
$f(-4) = -64 + 48 + 96 + 12 = 92$

$f(-1) = -1 + 3 + 24 + 12 = 38$

$f(2) = 8 + 12 - 48 + 12 = -16$



Problem #5: Sand is falling from a chute onto a pile that is shaped like a right circular cone at a rate of $48\pi ft^3/min$. If the radius of the pile is always 3 times the height, how fast is the height of the pile growing, when the height is 6 feet?



FOR A CONE, $V = \frac{\pi}{3} r^2 h$.

IF HEIGHT IS h, $r = 3h$

KNOW $\frac{dV}{dt} = 48\pi$.

WANT $\frac{dh}{dt}$ WHEN $h = 6$.

$$V = \frac{\pi}{3}(3h)^2 h = 3\pi h^3$$

$$\frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}$$

SO WHEN $h=6$, HAVE

$$48\pi = 9\pi \cdot 36 \cdot \frac{dh}{dt}$$

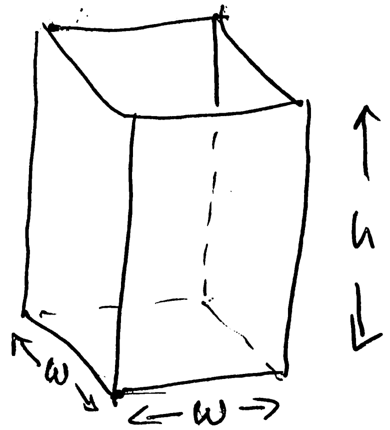
$$\frac{48\pi}{36 \cdot 9\pi} = \frac{dh}{dt}$$

$$\boxed{\frac{4}{27} = \frac{dh}{dt}}$$

ie GROWING AT $\frac{4}{27}$ FT/MIN.

Problem #6: An open-top box with a square base and rectangular sides is to have a volume of 9 ft^3 . The cost of the material to make the base is $\$2/ft^2$ and the cost of the material to make the sides is $\$3/ft^2$. Find the dimensions of the box that minimize the cost.

LET w BE ~~SIDE~~ WIDTH OF BASE, h BE THE HEIGHT.



THEN THE COST OF THE BOX IS

$$\underbrace{3(4wh)}_{\text{COST OF SIDES}} + \underbrace{2w^2}_{\text{COST OF BASE}}$$

$w > 0$

BUT, VOLUME IS 9, SO $9 = hw^2$

ie $h = 9/w^2$

$$\cancel{C(w) = 12\,9/9 + 2w^2 = \frac{4}{3}w^3 + 2w^2}$$

~~CRIT POINTS:~~

$$\cancel{C'(w) = 4w^2 + 2w = 2w(w+}$$

$$C(w) = 3\cdot 4\cdot \frac{9}{w^2}\cdot w + 2w^2 = 108/w + 2w^2.$$

$$C'(w) = -\frac{108}{w^2} + 4w.$$

$$C'(w) = 0 \iff 4w = \frac{108}{w^2} \iff w^3 = 27 \iff w = 3$$

CRIT POINTS AT $\underbrace{w=0}_{\text{IGNORE}}$ AND $w=3$.

$C''(w) = \frac{216}{w^3} + 4$, SINCE $C''(3) > 0$, $w = 3$ IS MIN.

COST IS MINIMAL FOR $3\times3\times1$ BOX.

Problem #7: Evaluate the following limits:

a) $\lim_{x\to 0}\frac{3\sin 4x}{2\tan 5x}$ $\left(\frac{0}{0}\right)$ SO CAN USE L'HÔPITALS.

$$= \lim_{x\to 0} \frac{12\cos(4x)}{10\sec^2(5x)} = \frac{12}{10} = \frac{6}{5}$$

b) $\lim_{x\to\infty}\frac{2x^3+4x^2-1}{6x^3+x-8}$ $= \lim_{x\to\infty} \frac{2x^3}{6x^3} = \lim_{x\to\infty} \frac{2}{6} = \frac{1}{3}.$

c) $\lim_{x\to 6}\frac{3x^2-12x-36}{x^2-x-30} = \lim_{x\to 6} \frac{3(x-6)(x+2)}{(x-6)(x+5)} = \lim \frac{3(x+2)}{(x+5)} = \frac{24}{11}$

OR, USE L'HÔPITAL'S TO GET

$$\lim_{x\to 6} \frac{6x-12}{2x-1} = \frac{36-12}{12-1} = \frac{24}{11}$$

JUST PLUG IN:

d) $\lim_{x \to 0} \frac{4e^{-x}}{5e^{x}+1}$ $= \frac{4}{5+1} = \frac{4}{6} = \frac{2}{3}$

e) $\lim_{h \to 0} \frac{(9+h)^2 - 81}{h}$ $= 18$ (DERIVATIVE OF x^2 AT $x=9$)

OR

$$\lim_{h \to 0} \frac{81 + 18h + h^2 - 81}{h} = \lim_{h \to 0} \frac{18h + h^2}{h} = \lim_{h \to 0} (18+h) = 18.$$

OR USE L'HÔPITAL'S:

$$\lim_{h \to 0} \frac{2(9+h)}{1} = 18$$