Name: FALL 2009 Id: SOLUTIONS

1. For each of the functions $f(x)$ given below, calculate the derivative $f'(x)$.

4 points (a) $f(x) = x^6 + \frac{x^2}{2} + \frac{1}{x}$

$6x^5 + x + \ln x$

4 points (b) $f(x) = x^3 e^{-x^2}$

$3x^2 e^{-x^2} - 2x^4 e^{-x^2}$

4 points (c) $f(x) = \arctan\sqrt{4x+1}$

$\frac{4}{1+(4x+1)} = \frac{2}{1+2x}$

4 points (d) $f(x) = \frac{(\sin 2x)^2 + (\cos 2x)^2}{e^{2x}} = \frac{1}{e^{2x}} = e^{-2x}$

$f'(x) = -2e^{-2x}$

(OR YOU CAN DO IT THE LONG WAY....)

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2. The derivative of a function $f(x)$ is

$$f'(x) = 3(x+3)(x-7)$$

4 points (a) On what intervals is $f(x)$ increasing?

CRITS @ $-3, 7$

$+ = (-)(-)$ -3 7

$(+)(-)$ $(+)(+)$

INCR | DECR | INCR

$\boxed{\text{INCREASING FOR } x < -3,\ x > 7}$

4 points (b) On what intervals is $f(x)$ concave up?

$f'(x) = 3(x^2 - 4x - 21)$

$f''(x) = 3(2x - 4)$

FOR $x < \frac{1}{2}$, $f''(x) < 0$

$x > 1/2$ $f''(x) > 0$

$\boxed{\text{CONC. UP FOR } x > 1/2}$

4 points (c) If $f(1) = 10$, what is $f(x)$?

SINCE $f'(x) = 3x^2 - 12x - 63$,

$f(x) = x^3 - 6x^2 - 63x + C$

WE KNOW $f(1) = 10$, SO

$10 = 1 - 6 - 63 + C \Rightarrow C = 78$

SO $\boxed{f(x) = x^3 - 6x^2 - 63x + 78.}$

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3. Compute each of the limits below. If a limit does not exist, please distinguish between $+\infty$, $-\infty$, and "no limiting behavior (DNE)". Give some justification or show some work for each of your answers.

3 points (a) $\lim_{x \to 1^+} \dfrac{\cos\left(\frac{\pi}{2}x\right)}{\ln x}$ $\frac{0}{0}$, USE L'HÔPITALS

$$= \lim_{x \to 1^+} \frac{-\frac{\pi}{2}\sin(\pi/2\, x)}{1/x} = \lim_{x \to 1^+} -x\frac{\pi}{2}\sin(\pi/2\, x) = -\pi/2.$$

3 points (b) $\lim_{x \to +\infty} \cos\left(\dfrac{1}{x}\right) = \cos(0) = 1$

3 points (c) $\lim_{x \to \infty} \left[\, \ln(3+x) - \ln(x-3) \,\right]$

$$= \lim_{x \to \infty}\left(\ln\left(\frac{3+x}{x-3}\right)\right) = \ln(1) = 0.$$

3 points (d) $\lim_{x \to 0} (1+3x)^{1/x} = L$

$$\text{so } \ln L = \lim_{x \to 0} \frac{1}{x}\ln(1+3x) \overset{\text{L'HOP}}{=} \lim_{x \to 0} \frac{\frac{3}{1+3x}}{1} = 3,$$

$$\therefore \boxed{L = e^3}$$

3 points (e) $\lim_{x \to +\infty} \dfrac{x^4 + 2x + 1}{3x^4 - 2x + 1}$

$$= \lim_{x \to \infty} \frac{x^4}{3x^4} = \frac{1}{3}$$

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10 points 4. Let

$$P(x) = 1 + 9\left(A\frac{x - x^2}{1 + x}\right)^{50} \qquad \text{where } A = (1 + \sqrt{2})^2.$$

For x between 0 and 1, the function $P(x)$ has an absolute maximum value of 10. Your goal is to accurately determine the value of x where this maximum occurs. **Write a number x in the box at right,** with $0 < x < 1$. $x =$ $\boxed{\sqrt{2} - 1}$

Your score on this problem will be equal to $P(x)$ evaluated at that number, and *rounded down* to an integer. Be very careful in your work: being off by even one tenth can change your score from 10 to 1. Note also that the only "partial credit" assigned will be determined by the number in the box, not by your work.

$$P'(x) = 9 \cdot 50\left(A\frac{x - x^2}{1+x}\right)^{49}\left(\frac{(1-2x)(1+x) - (x - x^2)}{(1+x)^2}\right) =$$

$$= 450\frac{\left(A\, x(1-x)\right)^{49}}{(1+x)^{51}}(1 - 2x - x^2)$$

CRITS @ $-1, 0, 1, \sqrt{2} \pm 1$

ONLY RELEVANT CRIT IS $\sqrt{2} - 1$, ITS A MAX

10 points 5. Use Newton's Method to find a solution of the equation

$$x^3 + x = 1 \qquad N(x) = x - \frac{x^3 + x - 1}{3x^2 + 1}$$

starting with an initial guess $x_1 = 0$. Write the next two approximations given by Newton's method (that is, x_2 and x_3).

If any of your x_i include fractions with a denominator bigger than 8, you made an error.

For ease of grading, write your answers here, with your work below:

$\boxed{x_2 = 1}$ $\boxed{x_3 = 3/4}$

$$X_2 = 0 - \frac{-1}{1} = 1$$

$$X_3 = 1 - \frac{1+1-1}{3+1} = \frac{3}{4}$$

6. The function $f(x)$ is defined as

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

4 points (a) Show that f is continuous at $x = 0$.

WHEN $x \neq 0$, $f(x) = x^2 \sin(1/x)$, CONTINUOUS SINCE POLY · TRIG.

AT $x=0$, MUST SHOW

$\lim_{x \to 0} f(x) = f(0)$ // $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin(\frac{1}{x})$

BUT $-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$

AND

$0 = \lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2$

BY SQUEEZE, LIMIT IS 0

4 points (b) Compute $f'(0)$. (Hint: use the definition of the derivative)

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = \lim_{h \to 0} h \sin(\tfrac{1}{h}) = 0.$$

4 points (c) Compute $f'(x)$ when $x \neq 0$.

IF $x \neq 0$, $f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$

4 points (d) Is $f'(x)$ a continuous function? Fully explain your answer.

NO, SINCE $\lim_{x \to 0} f'(x)$ DNE

BUT $f'(0) = 0$.

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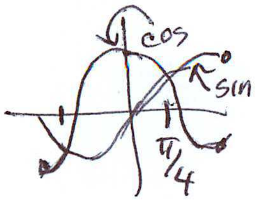
6 points 7. (a) Find all the critical numbers for the function

$$f(x) = \sin(2x) + \cos(2x) \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

and determine whether each is a relative minimum, relative maximum, or neither. You must justify your classification for full credit.

$f'(x) = 2\cos(2x) - 2\sin(2x)$.

$f'(x) = 0$ WHEN $\cos(2x) = \sin(2x)$. $\left(-\pi/2 \leq x \leq \pi/2\right)$



THERE ARE TWO SOLUTIONS:

$2x = \pi/4 \Rightarrow x = \pi/8$

$2x = -3\pi/4 \Rightarrow x = -3\pi/8$.

$f''(x) = -4\sin(2x) - 4\cos(2x)$

$f''(\pi/8) < 0$, SO $x = \pi/8$ IS REL MAX

$f''(-\frac{3\pi}{8}) > 0$, SO $x = -3\pi/8$ IS REL. MIN.

4 points (b) Find the absolute maximum and minimum values of $f(x)$ on the given interval.

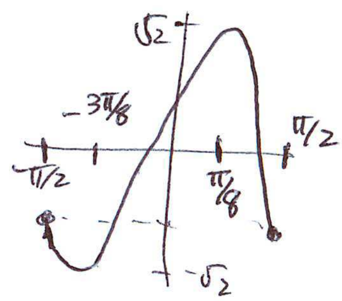
$f(-\pi/2) = \sin(-\pi) + \cos(-\pi) = -1$

$f(-3\pi/8) = \sin(\frac{-3\pi}{4}) + \cos(-3\pi/4) = -\sqrt{2}$

$f(\pi/8) = \sin(\pi/4) + \cos(\pi/4) = \sqrt{2}$

$f(\pi/2) = \sin(\pi) + \cos(\pi) = -1$

GRAPH IS



~~MAX~~ AT $(\pi/8, \sqrt{2})$

MIN AT $(-3\pi/8, -\sqrt{2})$.

8. Consider the curve given by $y^2 = x^3 + 3x^2$.

[5 points] (a) Write the equation of the line tangent to this curve at the point $(1, 2)$.

$2yy' = 3x^2 + 6x$

@(1,2): $4y' = 3 + 6$

$y' = 9/4$

TANGENT LINE IS

$y = 2 + \frac{9}{4}(x-1)$.

[3 points] (b) Using the result from the previous part, estimate the value of the y-coordinate when $x = 1.1$.

USING THE TANGENT LINE AT $x = 1.1 = 11/10$,

$y = 2 + \frac{9}{4}\left(\frac{1}{10}\right) = 2 + \frac{9}{40} = \frac{89}{40}$

[3 points] (c) For what (x, y) is the tangent line to the curve horizontal?

THIS IS WHEN $y' = 0$.

FROM (a): $y' = \frac{3x^2+6x}{2y} = \frac{3x(x+2)}{2y}$.

TROUBLE AT (0,0).

WHEN $x = -2$, $y^2 = 8 + 12$, $y = \pm\sqrt{20} = \pm 2\sqrt{5}$.

HORIZ TANGENTS AT $(-2, -2\sqrt{5})$ AND $(-2, +2\sqrt{5})$

10 points 9. On a recent mission to the International Space Station, a water balloon was filled at a constant rate of $128 \frac{cm^3}{sec}$. Because of the lack of gravity, the balloon remained a perfect sphere the entire time. At what rate was the radius of the balloon increasing[1] when the radius was $4\,cm$?

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 128$$

WANT $\frac{dr}{dt}$ WHEN $r = 4$.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

WHEN $r = 4$:

$$128 = 4\pi(4)^2 \frac{dr}{dt}$$

$$\frac{128}{64\pi} = \frac{dr}{dt}$$

$$\frac{2}{\pi} = \frac{dr}{dt}$$

[1]One or more of the following may be useful to you: The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, and its surface area is $4\pi r^2$. The weight of $1\,cm^3$ of water is 1 gram. Heidemarie Stefanyshyn-Piper managed to dodge the balloon when it was thrown at her, but lost her toolbag. Water balloons can be dangerous in space— only trained professionals should use them.