MATH 125 Solutions to Midterm 2 (Miracle Max)

1. For each of the functions $f(x)$ given below, find $f'(x)$).

$\boxed{\text{4 points}}$ (a) $f(x) = \dfrac{1+2x^2}{1+x^3}$

Solution: This is a straightforward quotient rule problem:

$$f'(x) = \frac{(4x)(1+x^3) - (1+2x^2)(3x^2)}{(1+x^3)^2} = \frac{4x - 3x^2 - 2x^4}{(1+x^3)^2}$$

The simplification is not required.

$\boxed{\text{4 points}}$ (b) $f(x) = \sin(4x)\tan(x)$

Solution: Apply the product rule, with a chain rule for the $\sin(4x)$ term to get

$$f'(x) = 4\cos(4x)\tan(x) + \sin(4x)\sec^2(x).$$

$\boxed{\text{4 points}}$ (c) $f(x) = \arctan\left(\sqrt{1+2x}\right)$

Solution: Applying the chain rule, we get

$$\frac{1}{1+\left(\sqrt{1+2x}\right)^2} \cdot \frac{1}{2}(1+2x)^{-1/2} \cdot (2) = \frac{1}{(2+2x)\sqrt{1+2x}}$$

2. Compute each of the following derivatives as indicated:

$\boxed{\text{4 points}}$ (a) $\dfrac{d}{du}\left[\dfrac{u^3}{5} + \dfrac{5}{u^3}\right]$

Solution: Write this as $\frac{1}{5}u^3 + 5u^{-3}$ and apply the power rule to get

$$\frac{3}{5}u^2 - 15u^{-4}.$$

$\boxed{\text{4 points}}$ (b) $\dfrac{d}{dx}\left[\, e^x - \pi^5 \,\right]$

Solution: Remember that π^5 is a constant and so its derivative is zero. Thus, we have $\dfrac{d}{dx}\left[\, e^x - \pi^5 \,\right] = e^x$.

4 points (c) $\dfrac{d}{dw}\left[\sqrt{1+\sqrt{1+w}}\right]$

Solution: View this as $\frac{d}{dw}\left[\left(1+(1+w)^{1/2}\right)^{1/2}\right]$ and apply the chain rule:

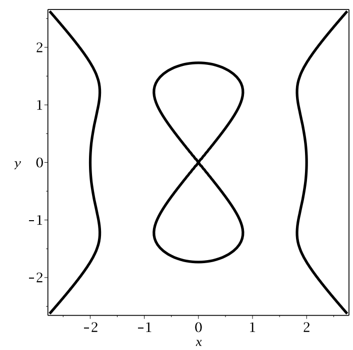
$$\frac{1}{2}\left(1+(1+w)^{1/2}\right)^{-\frac{1}{2}} \cdot \frac{1}{2}(1+w)^{-\frac{1}{2}} = \frac{1}{4\sqrt{1+w}\,\sqrt{1+\sqrt{1+w}}}$$

12 points 3. The set of points (x, y) which satisfy the relationship

$$y^2(y^2-3) = x^2(x^2-4)$$

lie on what is known as a "devil's curve", shown at right.

Write the equation of the line tangent to the given devil's curve at the point $(2, \sqrt{3})$.



Solution:

First, we use implicit differentiation to determine the slope of the tangent line. This will be slightly easier if we rewrite the equation as $y^4 - 3y^2 = x^4 - 4x^2$ first. Differentiating with respect to x gives

$$4y^3y' - 3 \cdot 2y \cdot y' = 4x^3 - 4 \cdot 2x \quad \text{and so} \quad y' = \frac{x(2x^2-4)}{y(2y^2-3)}.$$

At our desired point, $x = 2$ and $y = \sqrt{3}$, and so the slope is

$$y' = \frac{2 \cdot 4}{\sqrt{3} \cdot 3} = \frac{8}{3\sqrt{3}}.$$

This means the desired line is

$$y - \sqrt{3} = \frac{8}{3\sqrt{3}}(x-2).$$

4. Let $f(x) = x\ln(4x)$

4 points (a) Calculate $f'(x)$

Solution: Applying the product rule (and the chain rule) gives

$$f'(x) = \ln(4x) + x\frac{1}{4x}\cdot 4 = \ln(4x) + 1.$$

4 points (b) Calculate $f''(x)$

Solution: Taking the derivative of the above, we get $f''(x) = \dfrac{1}{x}$.

3 points (c) For what values of x is $f(x)$ increasing?

Solution: As we all know, $f(x)$ is increasing when $f'(x) > 0$. Thus, using our answer from part (a) tells us that we need to know when

$$\ln(4x) + 1 > 0 \qquad \text{or, equivalently,} \qquad \ln(4x) > -1.$$

Exponentiating both sides gives $4x > e^{-1}$, so we know that

$$f(x) \text{ is increasing for } x > \frac{1}{4e}.$$

3 points (d) For what values of x is $f(x)$ concave down?

Solution: We need to determine when $f''(x) < 0$. From part (b), this means

$$\frac{1}{x} < 0 \qquad \text{that is,} \qquad x < 0.$$

However, remember that $\ln(2x)$ is only defined for $x > 0$. Thus $f(x)$ is concave up for all values of x in its domain. There are no values of x where $f(x)$ is concave down.

12 points 5. Give the x and y coordinates of the (absolute) maximum and minimum values of the function

$$y = x^4 - 8x^2 + 1 \qquad \text{where} \qquad -1 \le x \le 3.$$

Solution: First, we locate the critical points. Since the function is a polynomial, $f'(x)$ is defined everywhere, so we only need concern ourselves with the x for which $f'(x) = 0$.

Since $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$, we have the critical points

$$x = 0 \qquad x = 2 \qquad x = -2$$

However, since we are concerned only with $-1 \leq x \leq 3$, we discard $x = -2$.

Now we evaluate f at each of the critical points, and the endpoints:

- $f(0) = 1$.
- $f(2) = 16 - 32 + 1 = -15$.
- $f(-1) = 1 - 8 + 1 = -6$.
- $f(3) = 81 - 72 + 1 = 10$.

The largest value of the above occurs at $x = 3$, $y = 10$. This is our absolute maximum.

The smallest occurs when $x = 2$ and $y = -15$, which is our absolute minimum.

12 points

6. Calvin's family is visiting a winery in Cutchogue, and he wanders off into the fermenting room and dives into one of the large cylindrical† wine vats. The vat has a diameter of 6 feet and is 8 feet tall. The vinter hears the splash and quickly opens the taps to drain the vat, which drains at a rate of 5 cubic feet per minute. How quickly is the height of wine in the tank dropping when the wine is 5 feet deep?

†The volume of a cylinder of height h and radius r is $\pi r^2 h$ and its surface area (excluding top and bottom) is $2\pi rh$. The density of the wine is about .98 kg/L or 61 pounds per cubic foot. 5 cubic feet is about 38 gallons or 142 liters. The wine is a rather sweet Riesling, but is probably less sweet after Calvin has been in it.

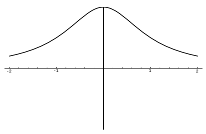
Solution: We have the formula for the volume of a cylinder $V = \pi r^2 h$. In our case, $r = 3$ since the diameter is 6, so we have $V = 9\pi h$ We want to know dh/dt.

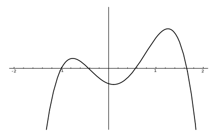
Since the vat is draining at a rate of 5 cubic feet per minute, we have $dV/dt = 5$.

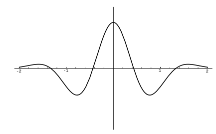
Using implicit differentiation, we get $\dfrac{dV}{dt} = 9\pi \dfrac{dh}{dt}$. So, we see that

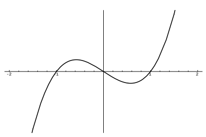
$$\frac{5}{9\pi} = \frac{dh}{dt}.$$

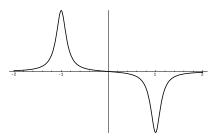
12 points 7. For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box. If the graph does not occur, use the letter **X**.

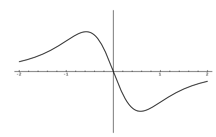
D 

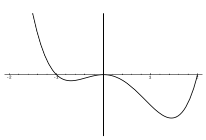
A: 

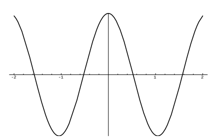
B: 

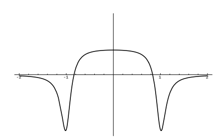
G 

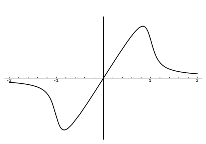
C: 

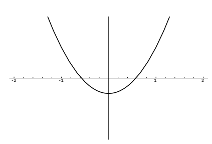
D: 

H 

E: 

F: 

F 

G: 

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