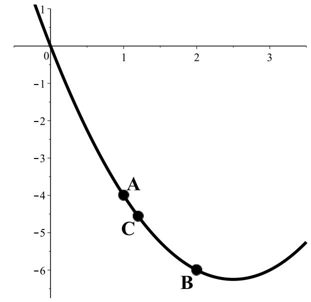
MAT 125 Solutions to Midterm 1 (Papaya)

1. Let $f(x) = x^2 - 5x$.

   We will use the three points $A = (1, f(1))$, $B = (2, f(2))$, and $C = (x, f(x))$ with x close to 1.



5 points (a) Calculate the slope of the line passing through A and B.

Solution: This is just the change in y over the change in x, so

$$\text{slope} = \frac{f(2) - f(1)}{2 - 1} = \frac{(4 - 10) - (1 - 5)}{1} = -2.$$

5 points (b) Give an equation for the line through A and B.

Solution: Any of the following is fine (since they are all the same line):

$$y - 4 = -2(x - 1), \quad y = -2 - 2x, \quad y + 6 = -2(x - 2).$$

5 points (c) Write an expression (depending on x) for the slope of the line through A and C.

Solution: Here we have

$$\text{slope} = \frac{f(x) - f(1)}{x - 1} = \frac{x^2 - 5x + 4}{x - 1} = \frac{(x - 1)(x - 4)}{x - 1}.$$

Note that the slope of the line through A and C is **not** $x - 4$, since it is not defined for $x = 1$ (ie, when $A = C$).

5 points (d) Calculate the slope of the line tangent to the graph of f at the point A.

Solution: We just take the limit:

$$\lim_{x \to 1} \frac{(x - 1)(x - 4)}{x - 1} = \lim_{x \to 1}(x - 4) = -3.$$

2. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

5 points (a) $\lim_{x\to 0} \frac{\sin x}{\tan x}$

Solution:

$$\lim_{x\to 0} \frac{\sin x}{\tan x} = \lim_{x\to 0} \frac{\sin x}{\frac{\sin x}{\cos x}} = \lim_{x\to 0} \cos x = \cos(0) = 1$$

5 points (b) $\lim_{x\to 1} \frac{x^2-1}{7x^2-7x}$

Solution:

$$\lim_{x\to 1} \frac{(x-1)(x+1)}{7x(x-1)} = \lim_{x\to 1} \frac{(x+1)}{7x} = \frac{1+1}{7} = \frac{2}{7}.$$

5 points (c) $\lim_{x\to 2} e^x \ln(x)$

Solution: For $x > 0$, $e^x \ln(x)$ is continuous, since it is the product of two continuous functions. Thus, the limit is what we get by evaluating the function at 2. Hence the limit is $e^2 \ln 2$.

5 points (d) $\lim_{x\to 1^+} \frac{x^2-9}{(x-3)(x-1)}$

Solution: Note for x close to 1, the numerator is close to -8 while the denominator tends towards zero. Thus, the function becomes unbounded at 1. Since we are looking at values of $x > 1$, the denominator is always positive. Hence, the limit is $-\infty$.

3. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

5 points (a) $\lim_{x\to +\infty} \frac{4x^2-1}{8x^2-27x+10}$

Solution: Since x is tending to infinity, the x^2 term dominates the others, so we have

$$\lim_{x\to +\infty} \frac{4x^2-1}{8x^2-27x+10} = \lim_{x\to +\infty} \frac{4x^2}{8x^2} = \lim_{x\to +\infty} \frac{4}{8} = \frac{1}{2}$$

5 points (b) $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

Solution:

$$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} 2x+h = 2x.$$

5 points (c) $\lim_{x\to 3} \frac{(x-3)^2}{|x-3|}$

Solution: Because of the absolute value, we need to consider $x < 3$ and $x > 3$ separately. For $x > 3$, $|x-3| = (x-3)$, and we get

$$\lim_{x\to 3^+} \frac{(x-3)^2}{|x-3|} = \lim_{x\to 3^+} \frac{(x-3)^2}{x-3} = \lim_{x\to 3^+} (x-3) = 0.$$

When $x < 3$, $|x-3| = -(x-3)$, yielding

$$\lim_{x\to 3^-} \frac{(x-3)^2}{|x-3|} = \lim_{x\to 3^-} \frac{(x-3)^2}{-(x-3)} = \lim_{x\to 3^-} (-x+3) = 0.$$

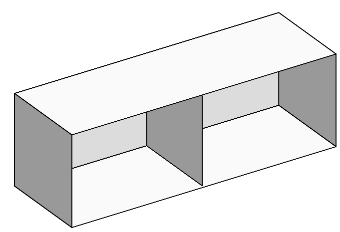
Since the two one-sided limits are the same, the overall limit is also zero.

5 points (d) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

Solution: We need to use the conjugate to deal with the square root:

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}+1}\right) = \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}.$$

15 points 4. An open, divided box is to be constructed from three square pieces of wood and three rectangular ones. The rectangular pieces will be used for the top, bottom and back, while the squares will form the ends and the divider.



Suppose the box is constructed from a total of 9 square feet of wood (and none is wasted). Let h represent the length of one of sides of the square pieces. Write an expression for the volume of the box in terms of h.

Solution:

The volume of the box is given by $V = \ell \cdot w \cdot h$, where ℓ, w, and h are the length, width, and height of the box, respectively.

First, since the ends are square, this means $w = h$, so $V = \ell h^2$.

Now, we only have 9 sq. ft. of wood to work with, so this will give us some relationship between ℓ and h. We have the box made from three square pieces that are $h \times h$, and three rectangular pieces that are $\ell \times h$. So,

$$9 = 3h^2 + 3\ell h$$

Solving for ℓ gives

$$\frac{3-h^2}{h} = \ell.$$

Plugging in to the formula $V = \ell h^2$ gives

$$V(h) = \frac{3-h^2}{h} \cdot h^2 = 3h - h^3$$

10 points 5. What value of k is necessary so that the function

$$f(x) = \begin{cases} k\tan\left(\frac{\pi}{4}x\right) & x < 1 \\ 3x^2 + x & x \geq 1 \end{cases}$$

is continuous for all positive values of x? Justify your answer fully.

Solution: The function f is continuous at all positive values of x except possibly at $x = 1$, since $\tan\left(\frac{\pi}{4}x\right)$ is continuous for $-4 < x < 4$ and $f(x)$ is a quadratic polynomial for $x \geq 1$.

Since $f(1) = 3 + 1 = 4 = \lim_{x\to 1^+} 3x^2 + x$, we need to find k so that $\lim_{x\to 1^-} f(x) = 4$.

Observe that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} k\tan\left(\frac{\pi}{4}x\right) = k,$$

so we need to take $k = 4$.

10 points 6. Write a limit that represents the slope of the line tangent to the graph of the function

$$f(x) = \begin{cases} |x-2|^{\sin(\pi x)} & x \neq 2 \\ 1 & x = 2 \end{cases}$$

at $x = 2$. You **do not need to evaluate the limit.**

Solution: We just use the definition of the derivative at $x = 2$:

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}.$$

Since h is not zero, $f(2+h) = |h|^{\sin\pi(2+h)}$ and $f(2) = 1$. So,

$$f'(2) = \lim_{h\to 0} \frac{|h|^{\sin\pi(2+h)} - 1}{h}.$$

If you prefer to use the version of the definition with $x \to 2$, you would have

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x\to 2} \frac{|x-2|^{\sin(\pi x)} - 1}{x-2}.$$

$\boxed{\text{16 points}}$ 7. Sketch the graph of a function $f(x)$ which satisfies all of the following properties:

- $f(2) = -1$
- $\lim_{x \to 2} f(x) \neq f(2)$
- $\lim_{x \to 1} f(x) = 1$
- $\lim_{x \to 0^+} f(x) = +\infty$
- $\lim_{x \to 0^-} f(x) = -\infty$
- $\lim_{x \to +\infty} f(x) = 0$
- $\lim_{x \to -\infty} f(x) = -\infty$
- $f'(-1) = 0$

Be sure to label all important values.

Solution: The graph below is one example, but there are several variations possible. The graph must pass through $(1, 1)$ and have a horizontal tangent at $x = -1$, and must have a removable discontinuity at $x = 2$ (with $f(2) = -1$), as well as a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$ on the right side.