Solutions to
Some problems of the type appropriate for
MAT125 Final Part I

1. Compute the following limits. Please distinguish between "$\lim f(x) = \infty$", "$\lim f(x) = -\infty$" and "limit does not exist even allowing for infinite values".

   (a) $\lim_{x \to -1} x^2 + x - 1$

   Solution: -1

   (b) $\lim_{x \to -3} \frac{x^2 + 2x - 3}{x + 3}$

   Solution: -4

   (c) $\lim_{t \to 0} \frac{\sqrt{2-t} - \sqrt{2}}{t}$

   Solution: $-\frac{1}{2\sqrt{2}}$

   (d) $\lim_{x \to \infty} \cos(1/x)$

   Solution: 1

   (e) $\lim_{x \to \infty} \frac{x^3 + 2x + 1}{x^3 - 2x + 1}$

   Solution: 1

   (f) $\lim_{x \to \pi/2} \frac{\sin x}{2x - \pi}$

   Solution: The limit does not exist.

   (g) $\lim_{x \to 0} \frac{\tan 3x}{2x}$

   Solution: $\frac{3}{2}$

2. For what value of k is the function

$$f(x) = \begin{cases} 3kx^2 + 4x + 1 & x < 1 \\ 2x^2 - 5kx - 1 & x \geq 1 \end{cases}$$

   continuous?

   Solution: $k = -\frac{1}{2}$

3. Compute the derivatives of the following functions

   (a) $f(x) = x^3 - 12x^2 + x + 2\pi$

   Solution: $f'(x) = 3x^2 - 24x + 1$

   (b) $f(x) = (2x+1)\sin(x)$

   Solution: $f'(x) = 2\sin(x) + (2x+1)\cos(x)$

   (c) $g(s) = \sqrt{1 + \ln(2s)}$

   Solution: $g'(s) = \dfrac{1}{2s\sqrt{1+\ln(2s)}}$

   (d) $h(t) = \dfrac{1+e^t}{1-e^t}$

   Solution: $h'(t) = \dfrac{e^t(1-e^t) + e^t(1+e^t)}{(1-e^t)^2} = \dfrac{2e^t}{(1-e^t)^2}$

   (e) $f(x) = (2x+2)^{10}$

   Solution: $f'(x) = 20(2x+2)^9$

   (f) $a(x) = \arctan(x^2)$

   Solution: $a'(x) = \dfrac{2x}{1+x^4}$

4. On what interval(s) is $f(x) = xe^{-x^2}$ increasing?

   Solution: $f(x)$ is increasing for $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$.

5. For what value(s) of x does $f(x) = x^3 + 3x^2 - 72x - 9$ have an inflection point?

   Solution: At $x = -1$.

6. Let $f(x) = -2x^3 + 6x^2 - 3$.
   (a) Compute f', f''.

   Solution: $f'(x) = -6x^2 + 12x = -6x(x-2)$ and $f''(x) = -12x + 12$.

   (b) On which intervals is $f(x)$ increasing/decreasing?

   Solution: $f(x)$ is increasing for $0 < x < 2$ and decreasing for $x < 0$ and $x > 2$.

   (c) On which intervals is $f(x)$ concave up/down?

   Solution: $f(x)$ is concave up for $x < 1$ and concave down for $x > 1$.

   (d) Find all critical points of $f(x)$. Which of them are local maximums? local minimums? neither? Justify your answer.

   Solution: The critical points are at $x = 0$ and $x = 2$. There is a local minimum at $x = 0$, since $f''(0) > 0$, and a local maximum at $x = 2$ since $f''(2) < 0$.

7. Stony Brook is going to build a new parking lot in the shape of a rectangle. It will be fenced in on three sides using 4000 feet of fence. The fourth side backs up to the woods and doesn't need a fence. What are the dimensions of the parking lot which has the maximum area?

   Solution: The desired field should have two sides of $1000'$ and one side of $2000'$ in order to maximize the area.

8. A sphere is expanding at a rate of 48 cubic inches per second. At what rate is the radius growing when the radius is $1/2$ inch?

   Solution: The radius is growing at $48/\pi$ inches per second.

9. Use differentials to approximate $\sqrt{9.02}$.

   Solution: $\sqrt{9.02} \approx 3 + .02/6 = 3 + 1/300$

10. Write the equation of the line tangent to the curve $y = \cos(2x)$ at $x = \pi/6$.

   Solution: $y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$

11. Find the value of $\frac{dy}{dx}$ when $x = -2$ and $y = 1$ if $\frac{4}{x^2} + y^4 = 2$.

   Solution: $\frac{dy}{dx} = -\frac{1}{4}$