MAT 125 Solutions to Second Midterm(papaya)

1. Compute each of the derivatives below as indicated.

4 points (a) $f(x) = 4x^8 - 5x^4 + 4x - e^3$.

Solution: $f'(x) = 32x^7 - 20x + 4$.
Don't forget that e^3 is a constant just a bit larger than 20, so its derivative is zero.

4 points (b) $f(x) = e^{3x} \tan x$

Solution: We need both the product rule and the chain rule here:

$$f'(x) = 3e^{3x} \tan x + e^{3x} \sec^2 x.$$

4 points (c) $f(x) = \dfrac{6x^3 - 5x}{\sec(\pi x) + x^2}$

Solution: By the quotient rule,

$$f'(x) = \frac{(18x^2 - 5)(\sec(\pi x) + x^2) - (6x^3 - 5x)(\pi \sec(\pi x) \tan(\pi x) + 2x)}{(\sec(\pi x) + x^2)^2}$$

This mess doesn't really simplify much, so why bother trying?

4 points (d) $f(x) = \arcsin(e^x)$

Solution: Using the chain rule, $f'(x) = \dfrac{e^x}{\sqrt{1 - e^{2x}}}$.

2. Calculate the indicated derivatives.

4 points (a) $\dfrac{d}{d\theta} \sin(3\theta) \cos(4\theta)$

Solution: By the product chain rules, we get $3\cos(3\theta)\cos(4\theta) - 4\sin(3\theta)\sin(4\theta)$.

4 points (b) Calculate the second derivative of $x^2 e^{2x}$ with respect to x.

Solution: If $f(x) = x^2 e^{2x}$, then using the product rule we get $f'(x) = 2xe^{2x} + 2x^2 e^{2x}$ or, if we want to factor it, $2(x + x^2)e^{2x}$. Thus,

$$f''(x) = 2(1 + 2x)e^{2x} + 4(x + x^2)e^{2x} = (2 + 8x + 4x^2)e^{2x}.$$

You could also leave it as $2e^{2x} + 4xe^{2x} + 8xe^{2x} + 4x^2 e^{2x}$.

4 points (c) $\dfrac{d^{10}}{dt^{10}} 11t^9$.

Solution: If you didn't immediately realize that the tenth derivative of a ninth degree polynomial is zero, maybe you'd see the pattern once you started taking derivatives:

$f(t) = 11t^9,\ f'(t) = 9{\cdot}11t^8,\ f''(t) = 8{\cdot}9{\cdot}11t^7,\ f'''(t) = 7{\cdot}8{\cdot}9{\cdot}11t^6,\ \ldots,\ f^{(9)}(t) = 1{\cdot}2{\cdot}3\cdots8{\cdot}9{\cdot}11.$

Since the 9^{th} derivative is a constant, $\frac{d^{10}}{dt^{10}} 11t^9 = 0.$

3. Let $f(x) = x\ln(x^8)$.

5 points (a) Calculate $f'(x)$.

Solution: Just apply the product rule and the chain rule to get

$$f'(x) = \ln(x^8) + \frac{x}{x^8} \cdot 8x^7 = \ln(x^8) + 8$$

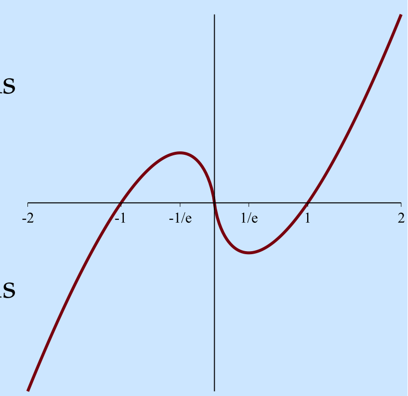
5 points (b) For what values of x is $f(x)$ decreasing? If there are none, write "NONE"; otherwise, describe *all* such x.

Solution: The function will be decreasing when $f'(x) < 0$. From the first part, we have $f'(x) = \ln(x^8) + 8$, but remember that $\ln(x^8) = 8\ln|x|$, so $f'(x) = 8(\ln|x| + 1)$. (Since the power of x inside the log is even, both $x < 0$ and $x > 0$ are in the domain, so we need to include the absolute value.)

Now, we want to know when $4(\ln|x| + 1) < 0$. This happens exactly when $\ln|x| < -1$. Exponentiating both sides gives

$$|x| < e^{-1} \quad \text{or} \quad -\frac{1}{e} < x < \frac{1}{e}.$$

At right is a graph of $f(x)$, and you can see that this corresponds to what we found.



$\boxed{\text{10 points}}$ 4. Find the slope of the line tangent to the curve $\sin(xy) = x^2 + y^2 - \pi$ at the point $(0, \sqrt{\pi})$.

Solution: To do this, we use implicit differentiation. Differentiating both sides with respect to x (remembering that y is some unknown function of x and using y' to represent the derivative of y with respect to x), we get

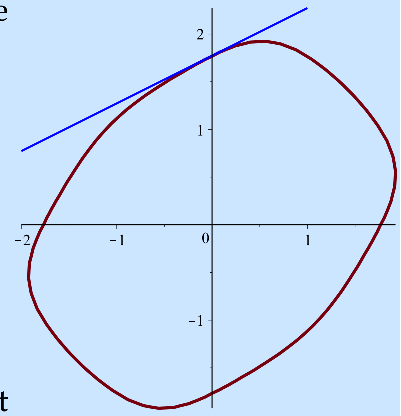
$$\cos(xy)(y + xy') = 2x + 2yy'$$

Now, we can subsitute $x = 0$ and $y = \sqrt{\pi}$, then solve for y'.

$$\cos(0)(\sqrt{\pi}) = 0 + 2\sqrt{\pi}y' \qquad \text{so} \quad 1/2 = y'.$$

Thus, the slope at the desired point is $1/2$.

The set of points satisfying $\sin(xy) = x^2 + y^2 - \pi$ is shown at right, together with the desired tangent line.



$\boxed{\text{10 points}}$ 5. Jimi Chiu makes "designer" shoes[1] that he sells at $250 a pair. He knows that the number of pairs of shoes he sells is a function of the price he charges; let's denote this by $N(p)$, where p is the price per pair. Market research tells him that $N'(250)$ is about -20; that is, if he raises the price by one dollar, he should expect to sell 20 fewer pairs. The amount of revenue $R(p)$ he makes at a given price will be given by $R(p) = p \cdot N(p)$.

If he typically sells 2000 pairs of shoes at $250 each, what is $R'(250)$? Should he raise the price a little?

Solution: $N(p)$ is some unknown function, but we know that $N(250) = 2000$ (since he sells 2000 pairs at a price of $250 each, and we know that $N'(250) = -20$.

We want to calculate $R'(250)$ where $R(p) = p \cdot N(p)$. By the product rule, we have $R'(p) = N(p) + pN'(p)$, so

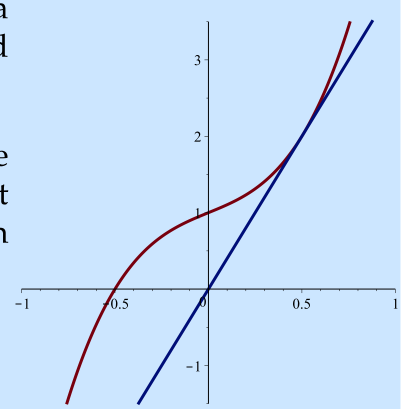
$$R'(250) = 2000 + 250 \cdot (-20) = 2000 - 5000 = -3000.$$

Since R' is negative, he will lose revenue if he raises the price. So he shouldn't charge more— in fact, he should lower the price to make more money.

[1] No relation to Jimmy Choo shoes, unless you don't look very closely. Mr. Chiu is also fond of Rollexx watches.

$\boxed{\text{10 points}}$ 6. Let $f(x) = 4x^3 + x + 1$. Find the equation of a line which passes through the origin and is also tangent to the curve $y = f(x)$ at some point (a, b)

Solution:
While you don't need the graph to do this problem, at right is a graph of what we are trying to solve, to help you keep in mind what is going on. We are looking for the blue line.



There are a couple of ways to do this problem; here's one. Note that since the line we are looking for goes through the origin, it is of the form $y = mx$. Since this line is also tangent to the graph of $f(x)$ at some point (a, b), we have

$$m = f'(a) = 12a^2 + 1$$

and $b = f(a) = 4a^3 + a + 1$.
So we have to find a so that

$$\begin{aligned} 4a^3 + a + 1 &= \left(12a^2 + 1\right)(a). \\ 4a^3 + a + 1 &= 12a^3 + a \\ 1 &= 8a^3. \end{aligned}$$

Thus, $a = 1/2$, and so $m = 12/4 + 1 = 4$.

The line we want is $y = 4x$.

10 points 7. Suppose $y = (1+\cos(x))^{(1+\sin(x))}$. What is $\frac{dy}{dx}$ when $x = \pi/2$ and $y = 1$?

Solution: To do this, we use logarithmic differentiation. Taking the log of both sides gives us

$$\ln(y) = \ln\left((1+\cos(x))^{(1+\sin(x))}\right) = (1+\sin x)\ln(1+\cos x),$$

and then differentiating both sides with respect to x gives us

$$\frac{y'}{y} = (\cos x)\ln(1+\cos x) + (1+\sin x)\cdot\frac{1}{1+\cos x}\cdot(-\sin x).$$

Substituting $x = \pi/2$ and $y = 1$ (remembering that $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$) yields

$$\frac{y'}{1} = 0\cdot\ln(1+0) + (1+1)\left(\frac{1}{1+0}\right)(-1) = 0 - 2 = -2.$$

A graph of the function with a tangent line at the desired point is shown below.