Math 125 Solutions to First Midterm

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points

(a) $\lim_{x \to 3} \dfrac{x^2 - 9}{6x(x-3)}$

Solution:

$$\lim_{x \to 3} \frac{x^2 - 9}{6x(x-3)} = \lim_{x \to 3} \frac{(x-3)(x+3)}{6x(x-3)} = \lim_{x \to 3} \frac{(x+3)}{6x} = \frac{3+3}{18} = \frac{1}{3}.$$

3 points

(b) $\lim_{x \to 4} e^x \ln(x)$

Solution: For $x > 0$, $e^x \ln(x)$ is continuous, since it is the product of two continuous functions. Thus, the limit is what we get by evaluating the function at 4. Hence the limit is $e^4 \ln 4$.

3 points

(c) $\lim_{x \searrow 2^+} \dfrac{x^2 - 9}{(x-2)}$

Solution: Note for x close to 2, the numerator is close to -5 while the denominator tends towards zero. Thus, the function becomes unbounded at 2. Since we are looking at values of $x > 2$, the denominator is always positive. Hence, the limit is $-\infty$.

2. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

3 points (a) $\lim_{x\to+\infty} \sqrt{16x^2+x}-4x$

Solution: This limit is of the form $\infty - \infty$, so we will have to combine the two terms to determine the limit. We can accomplish this by rationalizing the expression. Thus:

$$\begin{aligned}
\lim_{x\to+\infty} \sqrt{16x^2+x}-4x &= \lim_{x\to+\infty} \left(\sqrt{16x^2+x}-4x\right)\frac{\sqrt{16x^2+x}+4x}{\sqrt{16x^2+x}+4x} \\
&= \lim_{x\to+\infty} \frac{16x^2+x-16x^2}{\sqrt{16x^2+x}+4x} \\
&= \lim_{x\to+\infty} \frac{x}{\sqrt{16x^2+x}+4x} \\
&= \lim_{x\to+\infty} \frac{1/x}{1/x}\cdot\frac{x}{\sqrt{16x^2+x}+4x} \\
&= \lim_{x\to+\infty} \frac{1}{\sqrt{16+\frac{1}{x}}+4} = \frac{1}{\sqrt{16}+4} = \frac{1}{4+4} = \frac{1}{8}
\end{aligned}$$

3 points (b) $\lim_{h\to-3} \frac{(x+h)^2-x^2}{h}$

Solution:

$$\lim_{h\to-3} \frac{(x+h)^2-x^2}{h} = \lim_{h\to-3} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to-3} \frac{2xh+h^2}{h} = \lim_{h\to-3} 2x+h = 2x-3.$$

3 points (c) $\lim_{x\to0} \frac{\tan(\pi x)}{\sin(\pi x)}$

Solution: Since $\tan x = \frac{\sin x}{\cos x}$, we have

$$\frac{\tan(\pi x)}{\sin(\pi x)} = \frac{\sin(\pi x)}{\cos(\pi x)\sin(\pi x)} = \frac{1}{\cos(\pi x)},$$

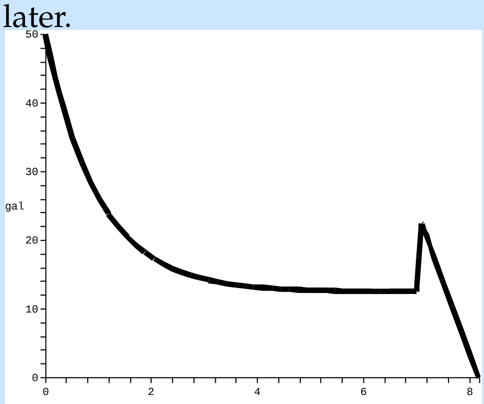
(where the last step is OK since $x \neq 0$). As $x \to 0$, $\cos(\pi x) \to \cos(0) = 1$. Thus, the limit is 1.

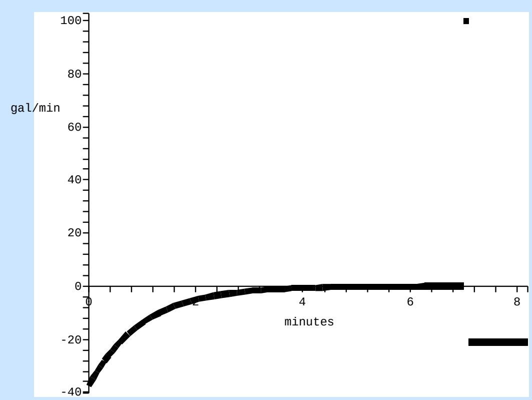
8 points 3. In the paragraph below is a description of how the amount of water $W(t)$ in a tub varied with time.

> The tub held about 50 gallons of green, brackish water, with some stuff floating in it that I didn't even want to guess about. I had to get it out of there. When I opened the drain the water drained out rapidly at first, but then it went slower and slower, until it stopped completely after about 5 minutes. The tub was about $1/4$-full of that nasty stuff. Would I have to stick my hand in it? *Ick*—there was no way I could do that. I just stared at it for a couple of minutes, but then I got an idea. I dumped in about 10 gallons of boiling water. That did something: there was this tremendous noise like *BLUUUUURP*, and then the tub drained steadily, emptying completely in just a minute or so.

Use this description to sketch a graph of $W(t)$ and its derivative $W'(t)$. Pay careful attention to slope and concavity. Label the axes, with units.

Solution: A pair of graphs something like those below agrees with the description (the graph of $W(t)$ is on the left, its derivative on the right). The graph starts out at 50, then decreases "slower and slower", (which is another way of saying it is decreasing and concave up) until it finally flattens out at about 5 minutes with a value of $12\frac{1}{2}$. The "spike" at around 7 minutes corresponds to when the 10 gallons of boiling water were added, raising the amount to $22\frac{1}{2}$, and then the level drops with constant slope, hitting the axis just about a minute later.





Of course, you might have minor variations. For example, the region around 7 minutes could be smooth, or discontinuous.

For the derivative, it should be negative everywhere except right around 7 minutes when the boiling water is added. The part before 7 minutes should be concave down and increasing, limiting on zero. In the version above, the short steep segment corresponds to adding the 10 gallons very quickly (at 100 gal/min), then the immediate quick draining is a jump back down to a fast rate of -20 gal/minute. If you made your graph of $W(t)$ smooth, the derivative should be continuous, with a quick bump going way up and then rapidly back down to -20 or so.

8 points

4. What value of k is necessary so that the function

$$f(x) = \begin{cases} kx + 4 & x < 3 \\ x^2 - x & x \geq 3 \end{cases}$$

is continuous for all values of x? Justify your answer fully.

Solution: The function f is continuous at all values except possibly at $x = 3$, since it is a line for $x < 3$ and a quadratic for $x \geq 3$.

Since $f(3) = 9 - 3 = 6 = \lim_{x \searrow 3^+} x^2 - x$, we need to choose k to ensure $\lim_{x \nearrow 3^-} f(x) = 6$ as well.

$$\lim_{x \nearrow 3^-} f(x) = \lim_{x \nearrow 3^-} kx + 4 = 3k + 4;$$

solving $3k + 4 = 6$ gives $k = 2/3$.

8 points

5. Write a limit that represents the slope of the graph

$$y = \begin{cases} |x|^{\sin x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

at $x = 0$. You **do not need to evaluate the limit.**

Solution: We just use the definition of the derivative at $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}.$$

Since h is not zero, $f(h) = |h|^{\sin h}$ and $f(0) = 1$. So,

$$f'(0) = \lim_{h \to 0} \frac{|h|^{\sin h} - 1}{h}.$$

If you prefer to use the version of the definition $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$, you get the same answer except with x instead of h. Whatever makes you happy.

(By the way, this is a tricky limit. If you can do it, you shouldn't be taking this class.)

6. Let $f(x) = 2x^2 - 5x + 5$.

3 points (a) Find the slope of the secant line passing through the points on the curve $y = f(x)$ where $x = 0$ and $x = 1$.

Solution: The slope of a line is the ratio of the change in y to the change in x. Here we have

$$\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{2 + 5}{1} = 7.$$

3 points (b) Find $f'(1)$.

Solution: Using the power rule, $f'(x) = 4x^2 - 5$, so $f'(1) = -1$.
If you didn't learn the power rule, you can do the limit instead:

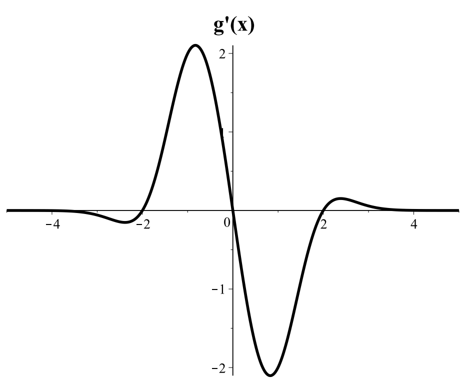
$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{(2x^2 - 5x + 5) - 2}{x - 1} = \lim_{x\to 1} \frac{2x^2 - 5x + 3}{x - 1}$$
$$= \lim_{x\to 1} \frac{(2x - 3)(x - 1)}{x - 1} = \lim_{x\to 1} 2x - 3 = -1$$

3 points (c) Write the equation of the tangent line to the graph of $y = f(x)$ when $x = 1$.

Solution: The point $(1, f(1))$ is on both the curve and the line. Now, $f(1) = 2 - 5 + 5 = 2$.
We just need the equation of the line of slope -1 passing through the point $(1, 2)$. This is

$$y - 2 = -1(x - 1) \qquad \text{or} \qquad y = -x + 3.$$

7. At right is the graph of **the derivative** $g'(x)$ of a function $g(x)$. Use it to answer each of the following questions.



List all values of x in the interval $[-5, 5]$ where $g(x)$ has a local maximum.

2 points (a) **Solution:** A local maximum for $g(x)$ will occur where $g'(x) = 0$ and $g'(x)$ changes from positive to negative. This happens when $x = 0$.

2 points (b) List all values of x in the interval $[-5, 5]$ where $g(x)$ has a local minimum.

Solution: We get a local minimum when $g'(x) = 0$ and $g'(x)$ changes from negative to positive. Thus, $x = -2$ or $x = 2$.

4 points (c) Assuming that the $g'(x)$ behaves the same for $x > 5$ as it does for $4 < x < 5$, which of the following should be true (circle your answer)?

A. $\lim_{x\to\infty} g(x) = +\infty$

B. $\lim_{x\to\infty} g(x)$ is a finite number

C. $\lim_{x\to\infty} g(x) = -\infty$

D. $\lim_{x\to\infty} g(x)$ does not exist

E. $\lim_{x\to\infty} g(x)$ can not be determined from this information

WHY? Justify your answer below. No credit without a justification.

Solution: The correct answer is "$\lim_{x\to\infty} g(x)$ is a finite number." In this graph, for $x > 4$, $g'(x)$ is zero. This means that the tangent line to $g(x)$ is horizontal for large x. Thus, the limit is a constant.

If you thought that the value of $g'(x)$ was just tending to zero, the correct answer is either B (if it tends to zero very fast), A (if it tends to zero slowly, but remains positive as it does so, or "cannot be determined".

Which answer gets full credit depends on your explanation, and whether your explanation matches your choice.

10 points 8. Sketch the graph of a function $f(x)$ which satisfies all of the following properties:

- $f(2) = 1$
- $\lim_{x\to 1} f(x) \neq f(1)$
- $\lim_{x\to 2} f(x) = 0$
- $\lim_{x\searrow 0^+} f(x) = -\infty$
- $\lim_{x\nearrow 0^-} f(x) = +\infty$
- $\lim_{x\to +\infty} f(x) = 0$
- $\lim_{x\to -\infty} f(x) = +\infty$
- $f'(-1) = 0$

Solution: Here is one graph that meets all of the criteria. There are many other graphs that also do the trick.