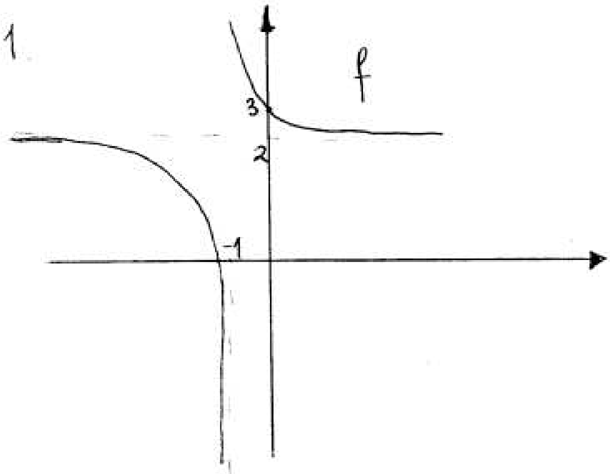
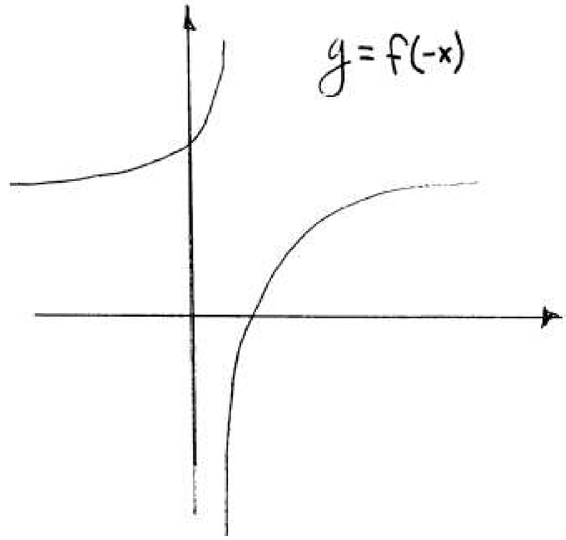
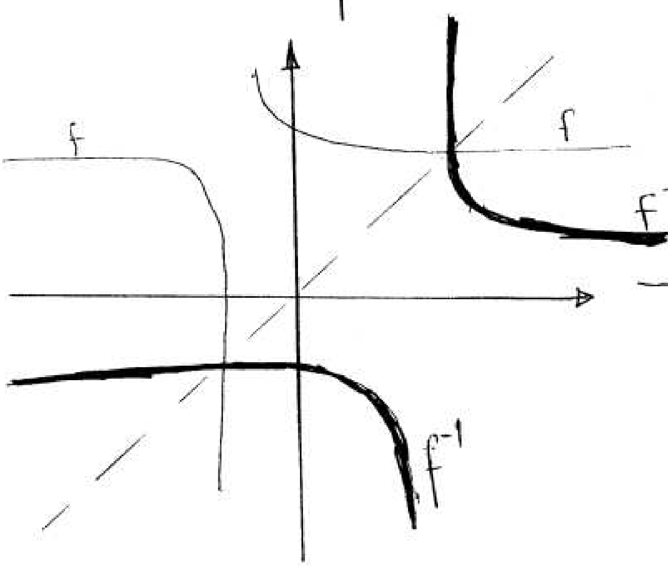
Sample First Exam
Solutions

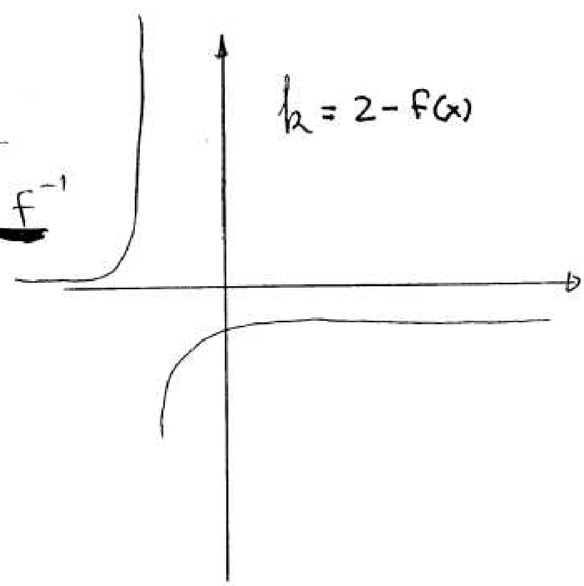
1.





The graph of g is obtained by reflecting the graph of f about the y-axis. For h reflect the graph of f about the line $y = x$. For k reflect the graph of f about the x-axis and then we shift 2 units upward.





② $h(x) = 6x^4 - 2x^2 + 1$

We may take $f(x) = 6x^2 - 2x + 1$ and $g(x) = x^2$ and then write $h(x) = f \circ g(x)$

This is not the only acceptable answer. We may also consider $f(x) = \frac{3}{2}x^2 - x + 1$ and $g(x) = 2x^2$ so $h(x) = f \circ g)(x)$.

③ - $f(0) = 1$ (f is defined in $x = 0$ and has value 1 for this x)

$\lim_{x \to 0} f(x)$ is not defined, because $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ exist but are different, as we shall see in a moment.

From the graph we can see that:

- $\lim_{x \to 0^+} f(x) = 2$; $\lim_{x \to 0^-} f(x) = 1$

-Even if $g(1) = -2$ the limit $\lim_{x \to 1} g(x)$ exists and is 0.

since $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x) = 0$

$-\lim_{x \to -2} (f(x) + g(x/2)) = \lim_{x \to -2} f(x) + \lim_{x \to -2} g(x/2) = 0 + (-2) = -2$

We used the fact that $\lim_{x \to -2} \frac{x}{2} = -1$

- We must calculate $\lim_{x \to 1} \frac{f(x)}{g(x)}$

From the graph we deduce that $\lim_{x \to 1} f(x) = 0$ and $\lim_{x \to 1} g(x) = 0$ so we can't know directly the limit.

$f(x) = -2x+2 \qquad g(x) = x-1$ (only for $x \neq 1$).

$$\Rightarrow \lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{-2x+2}{x-1} = \lim_{x\to 1} \frac{-2(x-1)}{x-1} = -2$$

$$- \lim_{x\to 3} (2f(x) - f(3)) = \lim_{x\to 3} 2f(x) - \lim_{x\to 3} f(3) = 2\lim_{x\to 3} f(x) - f(3)$$

$$= 2\cdot(-2) - 1 = -4+1 = -3.$$

From the graph we can see that we have possible problems only in 0 and 3. So we must check the continuity only in these points, all other points being points of continuity for f.

We've seen that $\lim_{x\to 0^+} f(x) = 2 \neq \lim_{x\to 0^-} f(x) = 1$ and $f(0) = 1$

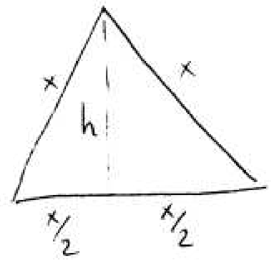
Also $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = -2$ but $f(3) = 1$

So we deduce that f is continuous from the left in 0 and f is not continuous in 3

4) Remaining area = Area of the circle − Area of the triangle, because the triangle is contained in the circle.

The circle has radius x, so its area is πx^2.

The triangle is equilateral, having all the edges of equal length x. Let's calculate its area



Applying Pythagora's theorem we have:

$$h^2 + \left(\frac{x}{2}\right)^2 = x^2 \Rightarrow h = \frac{x\sqrt{3}}{2}$$

So the area of the $\Delta = \frac{h \cdot x}{2} = \frac{x^2\sqrt{3}}{4}$

So our area is $A = \pi x^2 - \frac{x^2\sqrt{3}}{4} = x^2\left(\pi - \frac{\sqrt{3}}{4}\right)$

5) $f(x) = \frac{1}{x+2}$ $\quad g(x) = \frac{x-1}{x}$ The domains are:

$Dom(f) = \mathbb{R} \setminus \{-2\}$, $Dom(g) = \mathbb{R} \setminus \{0\}$

$$f \circ f\,(x) = \frac{1}{\frac{1}{x+2} + 2} = \frac{x+2}{2x+5} \qquad Dom(f \circ f) = \mathbb{R} \setminus \{-2, \tfrac{-5}{2}\}$$

$$(f \circ g)(x) = \frac{1}{\frac{x-1}{x} + 2} = \frac{x}{3x-1} \qquad Dom(f \circ g) = \mathbb{R} \setminus \{-2, \tfrac{1}{3}\}$$

$$(g \circ f)(x) = \frac{\frac{1}{x+2} - 1}{\frac{1}{x+2}} = -x-1 \qquad Dom(g \circ f) = \mathbb{R} \setminus \{0\}$$

$$(g \circ g)(x) = \frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}} = \frac{1}{1-x} \qquad Dom(g \circ g) = \mathbb{R} \setminus \{0, 1\}$$

(6)

a) $\lim\limits_{x\to 2} 3x^2 - x - 2 = 3\cdot 2^2 - 2 - 2 = 8$

b) $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2} = \lim\limits_{x\to 2} \dfrac{(x+3)(x-2)}{x-2} = \lim\limits_{x\to 2} x+3 = 5$

c) $\lim\limits_{q\to 2} \dfrac{2q^2+5}{q+2} = \dfrac{2\cdot 2^2+5}{2+2} = \dfrac{13}{4}$

d) We have $x^2+1 \geq f(x) \geq 4x-3$ for all x.

$\Rightarrow \lim\limits_{x\to 2} (x^2+1) \geq \lim\limits_{x\to 2} f(x) \geq \lim\limits_{x\to 2} (4x-3)$

But $\lim\limits_{x\to 2} (x^2+1) = 5$ and $\lim\limits_{x\to 2} 4x-3 = 5$.

Applying The Squeeze Theorem we have $\lim\limits_{x\to 2} f(x) = 5$.

e) $\lim\limits_{y\to -3} |y+3| = \cancel{\lim}\ 0$

f) $\lim\limits_{h\to 0} \dfrac{(2+h)^2-2}{h} = \lim\limits_{h\to 0} \dfrac{4+h^2+4h-2}{h} = \lim\limits_{h\to 0} \dfrac{h^2+4h-2}{h}$

We have $\lim\limits_{h\to 0} h^2+4h-2 = -2$ and $\lim\limits_{h\to 0} h = 0$

Since h could be positive or negative the limit is not defined.

Of course, what I **really** meant to ask was the following:

$$\lim_{h\to 2}\frac{(h+2)^2-4}{h}$$

To do this, we first try plugging in, and see that we get $\frac{0}{0}$, so its time to do more work. Notice that after squaring out $(h+2)^2$, we get

$$\begin{aligned}
\lim_{h\to 2}\frac{(h^2+4h+4)-4}{h} &= \lim_{h\to 2}\frac{h^2+4h}{h} \\
&= \lim_{h\to 2}\frac{h(h+4)}{h} \\
&= \lim_{h\to 2}(h+4) = 2+4 = 6
\end{aligned}$$

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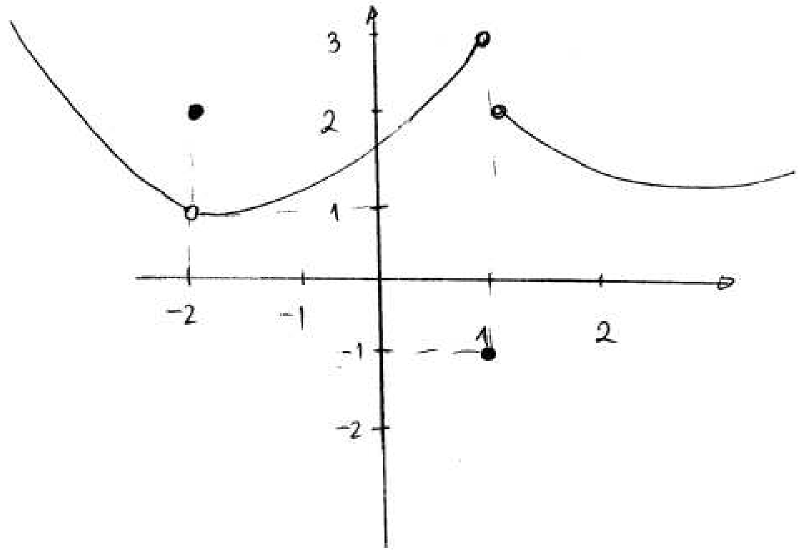

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- $\dfrac{8^{-1000}\sqrt{2^{10000}}}{16^{500}} = \dfrac{(2^3)^{-1000}\cdot 2^{\frac{10000}{2}}}{(2^4)^{500}} = \dfrac{2^{-3000}\cdot 2^{5000}}{2^{2000}} = 1$

- $\log_8 2 = \dfrac{\ln 2}{\ln 8} = \dfrac{\ln 2}{\ln 2^3} = \dfrac{\ln 2}{3\ln 2} = \dfrac{1}{3}$

  (OR, $\log_8 2$ IS THE SOLUTION TO $8^x = 2$. SINCE $8^{1/3} = 2$, $\log_8 2 = 1/3$)

- $\log_6 2 + \log_6 3 = \log_6 2\cdot 3 = 1$

- $\log_2\left(\frac{1}{16}\right) = \log_2 \frac{1}{2^4} = \log_2 2^{-4} = -4\log_2 2 = (-4)$

- $\ln e^{\pi} = \pi \ln e = \pi$

9. $f(x) = \dfrac{x+5}{3x-4}$

$\text{Dom} f = \mathbb{R} \setminus \{\frac{4}{3}\}$ Let $y = \dfrac{x+5}{3x-4} \iff 3xy - 4y = x+5$

$\iff 3xy - x = 4y+5 \iff x(3y-1) = 4y+5 \iff x = \dfrac{4y+5}{3y-1}$

So $f^{-1}(y) = \dfrac{4y+5}{3y-1}$ We observe that the inverse function f^{-1} is defined only if $3y-1 \neq 0$

That means $y \neq \frac{1}{3}$. So $\text{Dom}(f^{-1}) = \mathbb{R} \setminus \{\frac{1}{3}\}$

10. From hypothesis we have

$P(0) = 10000$ and $P(3) = 300 \cdot 1000$. Since $P(t) = ae^{kt}$ we must determine $\underline{a}$ and $\underline{k}$.

$P(0) = a \cdot e^{k \cdot 0} = a \Rightarrow a = 10000$

$P(3) = a \cdot e^{3k} = 300 \cdot 1000 \Rightarrow 10 \cdot 10\cancel{0}0 \cdot e^{3k} = 300 \cdot 10\cancel{0}0$

$\Rightarrow e^{3t} = 30$. Taking the logarithm $\Rightarrow 3k \ln e = \ln 30$

$\Rightarrow k = \dfrac{\ln 30}{3}$

The population when $t=4$ is $P(t) = ae^{k \cdot 4} = 10.000 \cdot e^{\frac{4}{3}\ln 30}$

$= 10000 \cdot e^{\ln (30)^{4/3}}$ Using the formula $e^{\ln x} = x \Rightarrow$

$P(4) = 10000 \cdot (30)^{4/3} = 10000 \cdot \sqrt[3]{(30)^4} = 10000 \cdot 30 \cdot \sqrt[3]{30}$

(11) Denote $f(x) = x^5 - 3x + 1$. We want to show that this function has a zero between 0 and 1.

$f(0) = 0 - 3 \cdot 0 + 1 = 1$

$f(1) = 1 - 3 + 1 = -1$

Since f is continuous we can apply the intermediate Value Theorem. Since 0 is between -1 and 1 there exist a number c such that $f(c) = 0$.