MAT 125 Solutions to Second Midterm, Vers. 2

1. For each of the functions $f(x)$ given below, find $f'(x)$).

   (a) $\boxed{\text{4 points}}$ $f(x) = x^5 + 5x^4 + 3x^2 + 9$

   Solution:
   $$f'(x) = 5x^4 + 20x^3 + 6x$$

   (b) $\boxed{\text{4 points}}$ $f(x) = x^4 e^x$

   Solution: This requires the product rule. Recall that the derivative of e^x is e^x.
   $$f'(x) = 4x^3 e^x + x^4 e^x$$

   (c) $\boxed{\text{4 points}}$ $f(x) = \dfrac{3x^2 + 2}{x^3 + 2\tan x}$

   Solution: Using the quotient rule,
   $$\frac{6x(x^3 + 2\tan x) - (3x^2 + 2)(x^2 + 2\sec^2 x)}{(x^3 + 2\tan x)^2}$$
   There is little point in trying to simplify this.

2. Compute each of the following derivatives as indicated:

   (a) $\boxed{\text{4 points}}$ $\dfrac{d}{d\theta}\left[\cos\left(\dfrac{\pi}{180}\theta\right)\right]$

   Solution: This is just the derivative of the $\cos\theta$, when θ is in degrees. Using the chain rule, we get
   $$-\frac{\pi}{180}\sin\left(\frac{\pi}{180}\theta\right)$$

   (b) $\boxed{\text{4 points}}$ $\dfrac{d}{du}\left[\sin(3u)\sin(4u)\right]$

   Solution: Use the product rule to get
   $$\left(\frac{d}{du}\sin(3u)\right)\sin(4u) + \sin(3u)\left(\frac{d}{du}\sin(4u)\right)$$
   and then use the chain rule to get the answer, which is
   $$3\cos(3u)\sin(4u) + 4\sin(3u)\cos(4u).$$

(c) $\boxed{\text{4 points}}$ $\dfrac{d}{dt}\left[\dfrac{t}{7} - \dfrac{7}{t}\right]$

Solution: If you rewrite this as $\frac{1}{7}t - 7t^{-1}$, it is clear the derivative is $\dfrac{1}{7} + 7t^{-2}$

3. $\boxed{\text{8 points}}$ Write a limit that represents the slope of the graph

$$y = \begin{cases} 6 + x\ln|x| & x \neq 0 \\ 6 & x = 0 \end{cases}$$

at $x = 0$. You **do not need to evaluate the limit.**

Solution: To do this, we need to remember the definition of the derivative, which is $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$. In the current case, $a = 0$, and notice that $f(0) = 6$, so we have

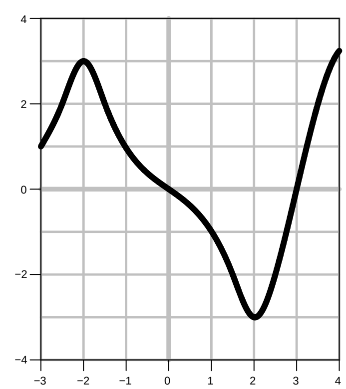
$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{(6 + h\ln|h|) - 6}{h}$$

This simplifies to

$$\lim_{h\to 0} \frac{h\ln|h|}{h} = \lim_{h\to 0} \ln|h| = -\infty,$$

although it wasn't required for you to do this.

4. At right is the graph of **the derivative** f' of a function.



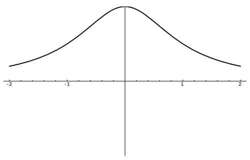
(a) $\boxed{\text{4 points}}$ List all values of x with $-3 \leq x \leq 4$ where $f(x)$ has a local minimum.

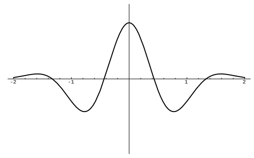
Solution: A local minimum for $f(x)$ will occur where $f'(x)$ changes from negative to positive. This happens at $x = 3$.

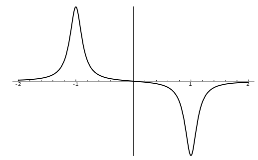
(b) $\boxed{\text{4 points}}$ At $x = -1$, is $f(x)$ concave up, concave down, or neither?

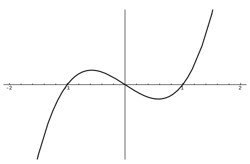
Solution: We know that a function is concave up when its second derivative is positive, and concave down when f'' is negative. The graph shows $f'(x)$, which is decreasing near $x = -1$. That means the derivative of $f'(x)$ is negative near $x = -1$, so $f''(-1) < 0$. Hence $f(x)$ is concave down at $x = -1$.

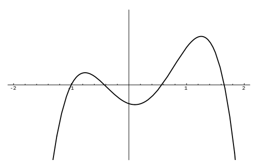
5. $\boxed{\text{16 points}}$ For each of the 4 functions graphed in the left column, find the corresponding derivative function among any of the 8 choices on the right (not just on the same row) and put its letter in the corresponding box.

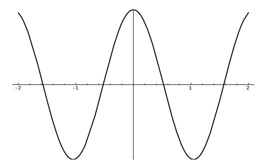
$\boxed{\textbf{F}}$ 

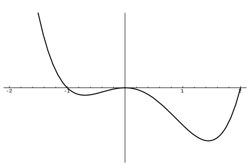
A: 

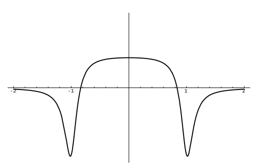
B: 

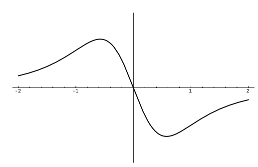
$\boxed{\textbf{H}}$ 

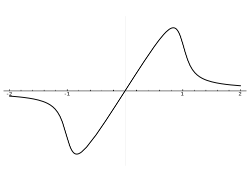
C: 

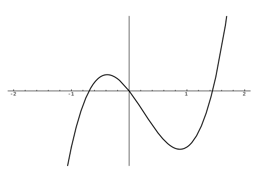
D: 

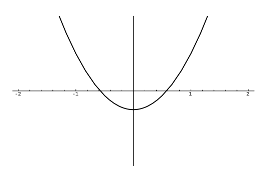
$\boxed{\textbf{G}}$ 

E: 

F: 

$\boxed{\textbf{E}}$ 

G: 

H: 

6. Let $f(x) = x\,e^{-2x}$.

   (a) $\boxed{\text{3 points}}$ Calculate $f'(x)$

   Solution: We use the product rule and the chain rule:

   $$f'(x) = e^{-2x} - 2x\,e^{-2x}$$

   (b) $\boxed{\text{3 points}}$ Calculate $f''(x)$?

   Solution: Taking the derivative of the above gives

   $$f''(x) = -2e^{-2x} - 2e^{-2x} + 4x\,e^{-2x}$$

   which simplifies to

   $$4x\,e^{-2x} - 4e^{-2x}$$

   (c) $\boxed{\text{4 points}}$ For what values of x is $f(x)$ increasing?

   Solution: To answer this, we need to know when $f'(x) > 0$, that is, where

   $$e^{-2x} - 2x\,e^{-2x} > 0$$

   Factoring out the exponential term gives $e^{-2x}(1 - 2x) > 0$, and since e^{-2x} is always positive, we only need ask where $1 - 2 > 0$. This happens for

   $$x < \frac{1}{2}.$$

   (d) $\boxed{\text{4 points}}$ For what values of x is $f(x)$ concave down?

   Solution: We need to know when $f''(x) < 0$, so factor $f''(x)$ as

   $$4e^{-2x}(2x - 1).$$

   As before, we can ignore the exponential term, since it is always positive, and we see that $f''(x) < 0$ when $x < 1/2$.

7. $\boxed{\text{10 points}}$ Write the equation of the line tangent to the curve

$$y = 3x^4 - 5x + \sqrt{x} \qquad \text{at } x = 1$$

Solution: To write the equation of a line, we need a point and a slope. Since the line is tangent to the curve at $x = 1$, it contains the point $(1, f(1)) = (1, -1)$.

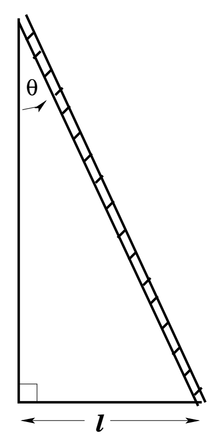
To get the slope, we calculate $f'(1)$. Taking the derivative gives

$$f'(x) = 12x^3 - 5 + \frac{1}{2}x^{-1/2},$$

so $f'(1) = 12 - 5 + \frac{1}{2} = \frac{15}{2}$. Hence the line is

$$y+1 = \frac{15}{2}(x-1), \quad \text{or, equivalently,} \quad y = \frac{15}{2}x - \frac{17}{2}$$

8. $\boxed{\text{10 points}}$ A ladder 14 feet long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, and let ℓ be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does ℓ change with respect to θ when $\theta = \frac{\pi}{6}$?



Solution: Since the ladder forms a right triangle with the wall, we have $\ell = 14\sin\theta$. The rate of change of ℓ with respect to θ is $\frac{d\ell}{d\theta}$, which is $14\cos\theta$. We want its value when $\theta = \frac{\pi}{6}$, so that is

$$14\cos\left(\frac{\pi}{6}\right) = 14 \cdot \frac{\sqrt{3}}{2} = 7\sqrt{3}$$