Math 125 Solutions to modified 2^{nd} Midterm

1. For each of the functions $f(x)$ given below, find $f'(x)$).

   (a) 4 points $f(x) = x^5 + 5x^4 + 4x^2 + 9$

   Solution:
   $$f'(x) = 5x^4 + 20x^3 + 8x$$

   (b) 4 points $f(x) = x^8 e^{2x}$

   Solution: This requires the product rule. Recall that the derivative of e^{2x} is $2e^{2x}$.
   $$f'(x) = 8x^7 e^{2x} + 2x^8 e^{2x}$$

   (c) 4 points $f(x) = \dfrac{3x^2 + 9}{x^3 + 2\tan x}$

   Solution: Using the quotient rule,
   $$\frac{6x(x^3 + 2\tan x) - (3x^2 + 9)(3x^2 + 2\sec^2 x)}{(x^3 + 2\tan x)^2}$$
   There is little point in trying to simplify this.

   (d) 4 points $f(x) = \arctan(\ln(x))$

   Solution: Using the chain rule, we have
   $$f'(x) = \frac{1}{1 + (\ln(x))^2} \cdot 1/x = \frac{1}{x + x(\ln x)^2}$$

2. Compute each of the following derivatives as indicated:

   (a) 4 points $\dfrac{d}{d\theta}\left[\cos\left(\dfrac{\pi}{180}\theta\right)\right]$

   Solution: This is just the derivative of the $\cos\theta$, when θ is in degrees. Using the chain rule, we get
   $$-\frac{\pi}{180}\sin\left(\frac{\pi}{180}\theta\right)$$

   (b) 4 points $\dfrac{d}{du}[\sin(3u)\sin(5u)]$

Solution: Use the product rule to get

$$\left(\frac{d}{du}\sin(3u)\right)\sin(5u) + \sin(3u)\left(\frac{d}{du}\sin(5u)\right)$$

and then use the chain rule to get the answer, which is

$$3\cos(3u)\sin(5u) + 5\sin(3u)\cos(5u).$$

(c) [4 points] $\frac{d}{dt}\left[\frac{t}{5} - \frac{5}{t}\right]$

Solution: If you rewrite this as $\frac{1}{5}t - 5t^{-1}$, it is clear the derivative is $\dfrac{1}{5} + 5t^{-2}$

3. Let $f(x) = x\,e^{-6x}$.

(a) [3 points] Calculate $f'(x)$

Solution: We use the product rule and the chain rule:

$$f'(x) = e^{-6x} - 6x\,e^{-6x}$$

(b) [3 points] Calculate $f''(x)$?

Solution: Taking the derivative of the above gives

$$f''(x) = -6e^{-6x} - 6e^{-6x} + 36x\,e^{-6x}$$

which simplifies to

$$36x\,e^{-6x} - 12e^{-6x}$$

(c) [4 points] For what values of x is $f(x)$ increasing?

Solution: To answer this, we need to know when $f'(x) > 0$, that is, where

$$e^{-6x} - 6x\,e^{-6x} > 0$$

Factoring out the exponential term gives $e^{-6x}(1 - 6x) > 0$, and since e^{-6x} is always positive, we only need ask where $1 - 6 > 0$. This happens for

$$x < \frac{1}{6}.$$

(d) 4 points For what values of x is $f(x)$ concave down?

Solution: We need to know when $f''(x) < 0$, so factor $f''(x)$ as

$$12e^{-6x}(3x - 1).$$

As before, we can ignore the exponential term, since it is always positive, and we see that $f''(x) < 0$ when $x < 1/3$.

4. 10 points Write the equation of the line tangent to the curve

$$y = 3x^4 - x + \sqrt{x} \qquad \text{at } x = 1$$

Solution: To write the equation of a line, we need a point and a slope. Since the line is tangent to the curve at $x = 1$, it contains the point $(1, f(1)) = (1, 3)$.

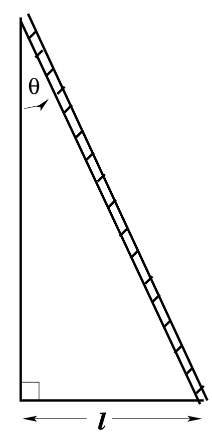
To get the slope, we calculate $f'(1)$. Taking the derivative gives

$$f'(x) = 12x^3 - 1 + \frac{1}{2}x^{-1/2},$$

so $f'(1) = 12 - 1 + \frac{1}{2} = \frac{23}{2}$. Hence the line is

$$y - 3 = \frac{23}{2}(x - 1), \quad \text{or, equivalently,} \quad y = \frac{23}{2}x - \frac{17}{2}$$

5. 10 points A ladder 12 feet long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, and let ℓ be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does ℓ change with respect to θ when $\theta = \frac{\pi}{6}$?



Solution: Since the ladder forms a right triangle with the wall, we have $\ell = 12\sin\theta$. The rate of change of ℓ with respect to θ is $\frac{d\ell}{d\theta}$, which is $12\cos\theta$. We want its value when $\theta = \frac{\pi}{6}$, so that is

$$12\cos\left(\frac{\pi}{6}\right) = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$$

6. (a) 8 points Write the equation of the line tangent to the curve

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

at the point $(0, -1/2)$.

Solution: Rather than solve for y directly (which is possible in this case, but tricky), we use implicit differentiation to figure out the slope of the tangent line. Differentiating both sides with respect to x (and remembering that y is a function of x) gives

$$2x + 2y\,y' = 2(2x^2 + 2y^2 - x)(4x + 4y\,y' - 1),$$

(on the right hand side we used the chain rule). Now we substitute $x = 0$ and $y = -1/2$ to obtain

$$0 - 1 \cdot y' = 2(0 + 2 \cdot \frac{1}{4} - 0)(0 + 4 \cdot \frac{-1}{2} \cdot y' - 1) \qquad \text{or} \qquad -y' = -2y' - 1$$

Solving the above for y' gives $y' = -1$.
Then the equation of the line with slope -1 passing through $(0, -1/2)$ is

$$y + \frac{1}{2} = -1(x - 0) \qquad \text{that is} \quad y = -x - \frac{1}{2}.$$

(b) 5 points Use your answer from the previous part to estimate the y-coordinate of a point on the curve with $x = 0.1$.

Solution: Plugging $x = 0.1$ to the tangent line found above gives us

$$y = -0.1 - 0.5 = -0.6$$

So, the point $(0.1, -0.6)$ should be close to a point on the curve.

If fact, the point on the lower part of the curve with $x = 1/10$ is

$$y = \frac{\sqrt{66 + 10\sqrt{45}}}{20} \approx 0.5768,$$

so our approximation isn't *too* bad.

7. 10 points If two resistors with resistance A and B are connected in parallel, the total resistance the total resistance R (in Ω) is given by the formula

$$\frac{1}{R} = \frac{1}{A} + \frac{1}{B}$$

If A is increasing at a rate of $0.3\,\Omega/s$ and B is decreasing at a rate of $0.2\,\Omega/s$, how fast is R changing when $A = 80\Omega$ and $B = 100\Omega$.

Solution: Translating the above, we have

$$\frac{dA}{dt} = 0.3, \qquad \frac{dB}{dt} = -0.2,$$

and we want to know dR/dt. Differentiationg the relationship with respect to t (and using the chain rule), we get

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{A^2}\frac{dA}{dt} - \frac{1}{B^2}\frac{dB}{dt} \tag{1}$$

We need to figure out what R is when $A = 80$ and $B = 100$, so we use

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{100}$$

to get $R = 400/9$.

Now we substitute into (1) above, and obtain

$$-\frac{1}{(400/9)^2}\frac{dR}{dt} = -\frac{1}{80^2}\cdot\frac{3}{10} - \frac{1}{100^2}\cdot(-\frac{2}{10})$$

to get

$$\frac{dR}{dt} = -\frac{43}{810} \approx -0.053\,\Omega/s.$$

(Sorry about the fractions. I took this one from the book without doing it first.)

8. For the function $f(x) = x^3 + 3x^2 - 24x$

(a) 4 points Calculate $f'(x)$.

Solution: $f'(x) = 3x^2 + 6x - 24$.

(b) 4 points At what points does $f(x)$ have a horizontal tangent line?

Solution: $f(x)$ will have a horizontal tangent when $f'(x) = 0$. Factoring gives

$$3x^2 + 6x - 24 = 3(x-2)(x+4),$$

so we have a horizontal tangent at $x = 2$ and $x = -4$.

(c) 6 points For $-3 \le x \le 3$, at which x values does $f(x)$ attain its maximum and minimum values?

Solution: The absolute maximum and minimum can occur only at the end-points or the critical numbers in the domain. Note that the critical point $x = -4$ is outside of the domain, so we must check at three places.

$f(-3) = -27+27+72 = 72 \quad f(2) = 8+12-48 = -28 \quad f(3) = 27+27-72 = -18$

So, for $-3 \le x \le 3$, we have

- the absolute maximum of $f(x)$ occurs at $x = -3,\ y = 72$
- the absolute minimum of $f(x)$ occurs at $x = 2,\ y = -28$