MAT 125 Solutions to Second Midterm, Vers. 2

1. For each of the functions $f(x)$ given below, find $f'(x)$).

$\boxed{\text{4 points}}$ (a) $f(x) = \dfrac{1+2x^2}{1+x^4}$

Solution: This is a straightforward quotient rule problem:

$$f'(x) = \frac{(4x)(1+x^4) - (1+2x^2)(4x^3)}{(1+x^4)^2} = \frac{4x - 4x^3 - 4x^5}{(1+x^4)^2}$$

The simplification is not required.

$\boxed{\text{4 points}}$ (b) $f(x) = \sin(2x)\cos(x)$

Solution: Apply the product rule, with a chain rule for the $\sin(2x)$ term to get

$$f'(x) = 2\cos(2x)\cos(x) - \sin(2x)\sin(x).$$

$\boxed{\text{4 points}}$ (c) $f(x) = \arctan\left(\sqrt{1+3x}\right)$

Solution: Applying the chain rule, we get

$$\frac{1}{1+\left(\sqrt{1+3x}\right)^2} \cdot \frac{1}{2}(1+3x)^{-1/2} \cdot (3) = \frac{3}{2(2+3x)\sqrt{1+3x}}$$

$\boxed{\text{4 points}}$ (d) $f(x) = \ln(\tan(x))$

Solution: Another chain rule problem:

$$f'(x) = \frac{1}{\tan(x)} \cdot \sec^2(x) = \frac{\cos(x)}{\sin(x)\cos^2(x)} = \sec(x)\csc(x).$$

2. Compute each of the following derivatives as indicated:

$\boxed{\text{4 points}}$ (a) $\dfrac{d}{dt}\left[\, e^{\sin^2(t)} \,\right]$

Solution: The chain rule gives

$$e^{\sin^2(t)} \cdot 2\sin(t) \cdot (-\cos(t)) = -2\sin(t)\cos(t)e^{\sin^2(t)}$$

4 points (b) $\dfrac{d}{du}\left[\, u^5 \ln(\sin(u)) \,\right]$

Solution: Using the product rule (and the chain rule), we obtain

$$5u^4 \ln(\sin(u)) + u^5 \frac{1}{\sin(u)} \cos(u) = u^4\big(5\ln(\sin(u)) + u\cot(u)\big)$$

4 points (c) $\dfrac{d}{dz}\left[\, \sqrt{1+\sqrt{1+z}} \,\right]$

Solution: View this as $\frac{d}{dz}\left[\, \left(1+(1+z)^{1/2}\right)^{1/2} \,\right]$ and apply the chain rule:

$$\frac{1}{2}\left(1+(1+z)^{1/2}\right)^{-\frac{1}{2}} \cdot \frac{1}{2}(1+z)^{-\frac{1}{2}} = \frac{1}{4\sqrt{1+z}\,\sqrt{1+\sqrt{1+z}}}$$

4 points (d) $\dfrac{d}{dx}\left[\, e^x - \pi^2 \,\right]$

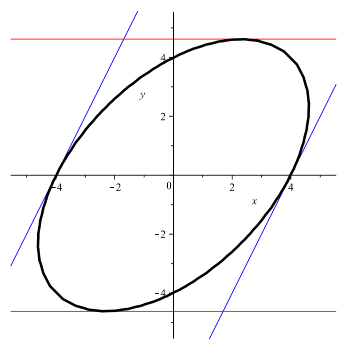
Solution: Remembering that π^2 is a constant, the derivative is just e^x.

3. The curve $x^2 - xy + y^2 = 16$ is an ellipse centered at the origin.

4 points (a) Find the points where this ellipse intersects the x-axis.

Solution: Since we are looking for points on the x-axis, this is where $y = 0$. Substituting $y = 0$ into the equation of the ellipse gives

$$x^2 = 16 \qquad \text{so} \qquad x = \pm 4.$$



6 points (b) Find the slope of the tangent line to this ellipse at each of the points from part (a).

Solution: Using implicit differentiation, we obtain $2x - \left(y + x\dfrac{dy}{dx}\right) + 2y\dfrac{dy}{dx} = 0$.
Substituting $y = 0$ and $x = \pm 4$ yields

$$\pm 8 = \pm 4\frac{dy}{dx}$$

and so the slope at either point is 2.

5 points

(c) Locate all points on this ellipse where the line tangent to the curve is horizontal.

Solution: To do this, we need to find all points (x, y) where the slope of the tangent line is zero. From part (b), we have

$$2x - \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0;$$

solving this for dy/dx gives

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}.$$

Thus, the slope of the tangent line will be zero when $y = 2x$.

Now we go back to the equation of the ellipse ($x^2 - xy + y^2 = 16$) and substitute $y = 2x$ to obtain

$$x^2 - x(2x) + (2x)^2 = 16, \qquad \text{or equivalently,} \qquad 3x^2 = 16.$$

Thus, $x = \pm 4/\sqrt{3}$. Since $y = 2x$, we have $y = \pm 8/\sqrt{3}$. Thus, the two points in question are

$$\left(\frac{4}{\sqrt{3}}, \frac{8}{\sqrt{3}}\right) \qquad \text{and} \qquad \left(-\frac{4}{\sqrt{3}}, -\frac{8}{\sqrt{3}}\right)$$

4. Let $f(x) = x\ln(2x)$

4 points

(a) Calculate $f'(x)$

Solution: Applying the product rule (and the chain rule) gives

$$f'(x) = \ln(2x) + x\frac{1}{2x} \cdot 2 = \ln(2x) + 1.$$

4 points

(b) Calculate $f''(x)$

Solution: Taking the derivative of the above, we get $f''(x) = \dfrac{1}{x}$.

3 points

(c) For what values of x is $f(x)$ increasing?

Solution: As we all know, $f(x)$ is increasing when $f'(x) > 0$. Thus, using our answer from part (a) tells us that we need to know when

$$\ln(2x) + 1 > 0 \qquad \text{or, equivalently,} \qquad \ln(2x) > -1.$$

Exponentiating both sides gives $2x > e^{-1}$, so we know that

$$f(x) \text{ is increasing for } x > \frac{1}{2e}.$$

3 points (d) For what values of x is $f(x)$ concave down?

Solution: We need to determine when $f''(x) < 0$. From part (b), this means

$$\frac{1}{x} < 0 \qquad \text{that is,} \qquad x < 0.$$

However, remember that $\ln(3x)$ is only defined for $x > 0$. Thus $f(x)$ is concave up for all values of x in its domain. There are no values of x where $f(x)$ is concave down.

12 points 5. The volume V of a spherical ball is growing at a constant rate of $1\ m^3/min$. Determine the rate of increase of its surface area S (in m^2/min) when its radius r is equal to 1 meter.

Perhaps you might find it helpful to recall that the volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$, and its surface area is $S = 4\pi r^2$.

Solution: The statement that the volume is growing at $1\frac{m^3}{min}$, we have $\frac{dV}{dt} = 1$. We are asked to find the rate of increase of the surface area when the radius is 1, that is, $\frac{dS}{dt}$ when $r = 1$.

We know that

$$V = \frac{4}{3}\pi r^3 \qquad \text{so} \qquad \frac{dV}{dt} = 4\pi r\frac{dr}{dt}$$

When $r = 1$, the equation on the right gives us $1 = 4\pi(1)\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{1}{4\pi}$.

Now we use

$$S = 4\pi r^2 \qquad \text{to get} \qquad \frac{dS}{dt} = 8\pi r\frac{dr}{dt}.$$

Since $r = 1$ and $\frac{dr}{dt} = \frac{1}{4\pi}$, we have

$$\frac{dS}{dt} = 8\pi\frac{1}{4\pi} = 2.$$

12 points 6. Use a linear approximation to estimate the value of $\arcsin(.52)$

Solution: We use the following two facts:

- $f(x) \approx f(a) + f'(a)(x-a)$ for x near a,
- $\arcsin(.5) = \pi/6$.

Thus, if we take $a = \frac{1}{2}$ and $f(x) = \arcsin(x)$, we can approximate $f(.52)$ using the tangent line.

Recalling that $f'(a) = \dfrac{1}{\sqrt{1-a^2}}$, we have

$$f'(1/2) = \frac{1}{\sqrt{1-(1/2)^2}} = \frac{1}{\sqrt{3/4}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}.$$

Thus, we have

$$\arcsin(.52) \approx \frac{\pi}{6} + \frac{2}{\sqrt{3}}(.52 - .5) = \frac{\pi}{6} + \frac{0.04}{\sqrt{3}}.$$

If you prefer to phrase this in terms of differentials, you get the same answer. The differential of $\arcsin(x)$ is $dy = \dfrac{dx}{\sqrt{1-x^2}}$. Taking $x = \frac{1}{2}$ and $dx = .02$, we have

$$\arcsin(.52) \approx \arcsin(1/2) + dy = \frac{\pi}{6} + \frac{0.04}{\sqrt{3}}.$$

This is approximately $\frac{\pi}{6} + 0.023094$ while $\arcsin(.52)$ is $\frac{\pi}{6} + 0.023252$ to 6 places. Obviously, you wouldn't have been able to determine that without a calculator.

Note that the function $\arcsin(x)$ gives a result in radians. If you gave an answer in degrees, I suspect that you got the derivative all wrong... that is, you neglected to adjust by $180/\pi$.