MATH 125 Solutions to Second Midterm, Vers. 1

1. For each of the functions $f(x)$ given below, find $f'(x)$).

   (a) $\boxed{\text{3 points}}$ $f(x) = x^9 + 5x^4 + 2x^2 + \pi^2$

   Solution: Don't forget that $\pi^2 \approx 9.87$, so its derivative is 0.

   $$f'(x) = 9x^8 + 20x^3 + 4x$$

   (b) $\boxed{\text{3 points}}$ $f(x) = \cos(x)\sin(4x)$

   Solution: This requires the product rule, and the chain rule.

   $$f'(x) = -\sin(x)\sin(4x) + 4\cos(x)\cos(4x)$$

   (c) $\boxed{\text{3 points}}$ $f(x) = \dfrac{\sin(x)}{\cos(x)}$

   Solution: After simplifying $\frac{\sin(x)}{\cos(x)} = \tan(x)$, we just remember that the derivative of $\tan(x)$ is $\sec^2(x)$.
   Alternatively, if you prefer to use the quotient rule, you should get

   $$\frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

   (d) $\boxed{\text{3 points}}$ $f(x) = \arctan\left(x^2\right)$

   Solution: This is a straight chain-rule problem.

   $$f'(x) = \frac{1}{1 + (x^2)^2} \cdot (2x) = \frac{2x}{1 + x^4}$$

   Several people were confused and thought that $\arctan(x) = \frac{1}{\tan(x)} = \cot(x)$; this is nonsense. You should know that $\arctan(x) = y$ means that $\tan(y) = x$.

2. Compute each of the following derivatives as indicated:

   (a) $\boxed{\text{3 points}}$ $\dfrac{d}{dt}\left[\dfrac{e^t - e^{-t}}{e^t + e^{-t}}\right]$

Solution: Applying the quotient rule gives us

$$\frac{(e^t+e^{-t})(e^t+e^{-t})-(e^t-e^{-t})(e^t-e^{-t})}{(e^t+e^{-t})^2} = \frac{(e^{2t}+2+e^{-2t})-(e^{2t}-2+e^{-2t})}{(e^t+e^{-t})^2} = \frac{4}{(e^t+e^{-t})^2}$$

(b) $\boxed{\text{3 points}}$ $\dfrac{d}{du}\left[\, u\ln(u)\,\right]$

Solution: Using the product rule, we have

$$1\cdot\ln(u)+u\cdot\frac{1}{u}=\ln(u)+1$$

(c) $\boxed{\text{3 points}}$ $\dfrac{d}{dz}\left[\, \ln(\sec(3z))\,\right]$

Solution: From the chain rule, we have

$$\frac{1}{\sec(3z)}\cdot\sec(3z)\tan(3z)\cdot 3 = 3\tan(3z)$$

(d) $\boxed{\text{3 points}}$ $\dfrac{d}{dx}\left[\, e^x - x^e\,\right]$

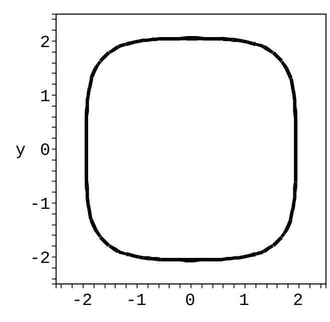
Solution: Remembering that e is a constant, we have $e^x - e\,x^{e-1}$.

3. $\boxed{\text{10 points}}$ Let $\mathcal{C}$ be the curve which consists of the set of points for which

$$x^4+x^2+y^4=18$$

(see the graph at right).

Write the equation of the line tangent to $\mathcal{C}$ which passes through the point $(1,-2)$.



Solution: In order to write the equation of a line, we need a point on the line (which we have: $(1,-2)$) and the slope of the line. For the slope, we need dy/dx at the given point.

We *could* solve for y, getting $y=\pm\sqrt[4]{18-x^4-x^2}$, and take the derivative of the resulting function to get $y'=\pm(18-x^4-x^2)^{-3/4}(-4x^3-x)$.

Instead, let's use implicit differentiation:

$$4x^3 + 2x + 4y^3 \frac{dy}{dx} = 0$$

Since we want the slope when $x = 1$ and $y = -2$, we plug in and solve for dy/dx.

$$4 + 2 - 32 \frac{dy}{dx} = 0, \qquad \text{so} \qquad \frac{dy}{dx} = \frac{-6}{-32} = \frac{3}{16}$$

Thus, the desired line is

$$y + 2 = \frac{3}{16}(x - 1) \qquad \text{or} \qquad y = \frac{3}{16}x - \frac{35}{16}$$

4. $\boxed{\text{10 points}}$ Give the x and y coordinates of the (absolute) maximum and minimum values of the function

$$y = x^4 - 8x^2 + 1 \qquad \text{where} \qquad -1 \leq x \leq 3.$$

Solution: First, we locate the critical points. Since the function is a polynomial, $f'(x)$ is defined everywhere, so we only need concern ourselves with the x for which $f'(x) = 0$.

Since $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$, we have the critical points

$$x = 0 \qquad x = 2 \qquad x = -2$$

However, since we are concerned only with $-1 \leq x \leq 3$, we discard $x = -2$.

Now we evaluate f at each of the critical points, and the endpoints:

- $f(0) = 1$.
- $f(2) = 16 - 32 + 1 = -15$.
- $f(-1) = 1 - 8 + 1 = -6$.
- $f(3) = 81 - 72 + 1 = 10$.

The largest value of the above occurs at $x = 3$, $y = 10$. This is our absolute maximum.

The smallest occurs when $x = 2$ and $y = -15$, which is our absolute minimum.

5. Let $f(x) = x\,e^{-4x}$.

   (a) $\boxed{\text{3 points}}$ Calculate $f'(x)$

   Solution: We use the product rule and the chain rule:

   $$f'(x) = e^{-4x} - 4x\,e^{-4x}$$

   (b) $\boxed{\text{3 points}}$ Calculate $f''(x)$

   Solution: Taking the derivative of the above gives

   $$f''(x) = -4e^{-4x} - 4e^{-4x} + 16x\,e^{-4x}$$

   which simplifies to

   $$16x\,e^{-4x} - 8e^{-4x}$$

   (c) $\boxed{\text{3 points}}$ For what values of x is $f(x)$ increasing?

   Solution: To answer this, we need to know when $f'(x) > 0$, that is, where

   $$e^{-4x} - 4x\,e^{-4x} > 0$$

   Factoring out the exponential term gives $e^{-4x}(1 - 4x) > 0$, and since e^{-4x} is always positive, we only need ask where $1 - 4x > 0$. This happens for

   $$x < \frac{1}{4}.$$

   (d) $\boxed{\text{3 points}}$ For what values of x is $f(x)$ concave down?

   Solution: We need to know when $f''(x) < 0$, so factor $f''(x)$ as

   $$8e^{-4x}(2x - 1).$$

   As before, we can ignore the exponential term, since it is always positive, and we see that $f''(x) < 0$ when $x < 1/2$.

6. $\boxed{\text{10 points}}$ A leaky oil tanker is anchored offshore. Because the water is very calm, the oil slick always stays circular as it expands, with a uniform depth of 1 meter. If the oil is leaking from the tanker at a rate of 100 $\frac{m^3}{hr}$, how fast is the radius of the slick expanding (in $\frac{m}{hr}$) when the diameter is 20 meters?

Solution: First, notice that since we are given the rate of oil leaking out from the tanker, this is the rate of change of volume of oil ($dV/dt = 100\frac{m^3}{hr}$), and we want to know the rate of change of the radius (dr/dt). This means we need to write a formula for the volume of the oil as a function of the radius.

Since we are told that the oil slick is circular and has a constant depth of 1 meter, it is a cylinder of height 1 and radius r. That is,

$$V = \pi r^2$$

Taking the derivative with respect to time gives

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt}$$

and plugging in the given values yields

$$100 = 2\pi \cdot 10 \frac{dr}{dt},$$

so we have

$$\frac{dr}{dt} = \frac{100}{20\pi} = \frac{5}{\pi}.$$