MAT 125 Solutions to First Midterm

1. Compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

$\boxed{\text{3 points}}$ (a) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{7x(x-1)}$

Solution:

$$\lim_{x\to 1} \frac{(x-1)(x+1)}{7x(x-1)} = \lim_{x\to 1} \frac{(x+1)}{7x} = \frac{1+1}{7} = \frac{2}{7}.$$

$\boxed{\text{3 points}}$ (b) $\displaystyle\lim_{x\to\infty} 4\cos\left(\frac{\pi}{x}\right)$

Solution:

$$\lim_{x\to\infty} 2\cos(\pi/x) = 2\cos(0) = 2.$$

$\boxed{\text{3 points}}$ (c) $\displaystyle\lim_{x\to 1} \frac{x^2}{(x-1)^2}$

Solution: Note for x close to 1, the numerator is close to 1 while the denominator tends towards zero. Thus, the function becomes unbounded at 1. Note also that the denominator is always positive. Hence, the limit is $+\infty$.

2. More of the same: compute each of the following limits. If the limit is not a finite number, please distinguish between $+\infty$, $-\infty$, and a limit which does not exist (DNE). Justify your answer, at least a little bit.

$\boxed{\text{3 points}}$ (a) $\displaystyle\lim_{x\to\infty} \frac{x^2-4}{7x(x-2)}$

Solution: For x very large, $x^2-4 \approx x^2$, and $x-2 \approx x$. Thus

$$\lim_{x\to\infty} \frac{x^2-4}{7x(x-2)} = \lim_{x\to\infty} \frac{x^2}{7x(x)} = \lim_{x\to\infty} \frac{1}{7} = \frac{1}{7}$$

$\boxed{\text{3 points}}$ (b) $\displaystyle\lim_{h\to 1} \frac{(x+h)^2-x^2}{h}$

Solution:

$$\lim_{h\to 1} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 1} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 1} \frac{2xh+h^2}{h} = \lim_{h\to 1} 2x+h = 2x+1.$$

3 points (c) $\lim_{x \to -\infty} e^x \cos(x)$

Solution: Observe that for any x, we have $-1 \leq \cos(x) \leq 1$, and so we also have $-e^x \leq e^x \cos(x) \leq e^x$. Applying the squeeze theorem,

$$\lim_{x \to -\infty} (-e^x) \leq \lim_{x \to -\infty} e^x \cos(x) \leq \lim_{x \to -\infty} (e^x),$$

that is,

$$0 \leq \lim_{x \to -\infty} e^x \cos(x) \leq 0.$$

Hence, the limit is 0.

3. Let $f(x) = 4x^3 - 7x + 2$.

3 points (a) Find the slope of the secant line passing through the points on the curve $y = f(x)$ where $x = 0$ and $x = 1$.

Solution: The slope of a line is the ratio of the change in y to the change in x. Here we have

$$\text{slope} = \frac{f(1) - f(0)}{1 - 0} = \frac{-1 - 2}{1} = -3.$$

3 points (b) Find $f'(1)$.

Solution: Using the power rule, $f'(x) = 12x^2 - 7$, so $f'(1) = 5$.

3 points (c) Write the equation of the tangent line to the graph of $y = f(x)$ when $x = 1$.

Solution: The point $(1, f(1))$ is on both the curve and the line. Now, $f(1) = 4 - 7 = -3$. We just need the equation of the line of slope 5 passing through the point $(1, -3)$. This is

$$y + 3 = 5(x - 1) \qquad \text{or} \qquad y = 5x - 8.$$

3 points (d) At $x = 1$, is $f(x)$ concave up, concave down, or neither? Justify your answer fully.

Solution: Since $f''(x) = 24x$, we know $f''(1) > 0$. Thus $f(x)$ is concave up at $x = 1$.

8 points 4. For what values of x is the function $f(x) = \dfrac{e^x}{3 - e^{1/x}}$ continuous?

Solution: Since $f(x)$ is a composition of exponentials and rational functions, it is continuous everywhere on its domain.

Since $1/x$ is not defined for $x = 0$, the function is not continuous there.

Furthermore, there will be a discontinuity when the denominator is zero. That is, where $3 - e^{1/x} = 0$, or

$$\begin{aligned} 3 &= e^{1/x} \\ \ln(3) &= \ln\left(e^{1/x}\right) = 1/x \\ x &= \frac{1}{\ln(3)}. \end{aligned}$$

Thus, $f(x)$ is continuous at all real numbers except $x = 0$ and $x = \frac{1}{\ln(3)}$.

8 points 5. Write a limit that represents the slope of the graph

$$y = \begin{cases} |x|^x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

at $x = 0$. You **do not need to evaluate the limit.**

Solution: We just use the definition of the derivative at $x = 0$:

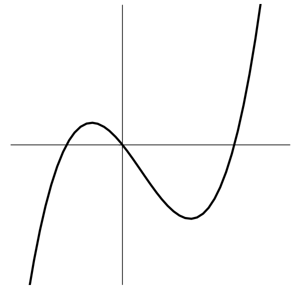
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}.$$

Since h is not zero, $f(h) = |h|^h$ and $f(0) = 1$. So,

$$f'(0) = \lim_{h \to 0} \frac{|h|^h - 1}{h}.$$

If you prefer to use the version of the definition $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$, you get the same answer except with x instead of h.

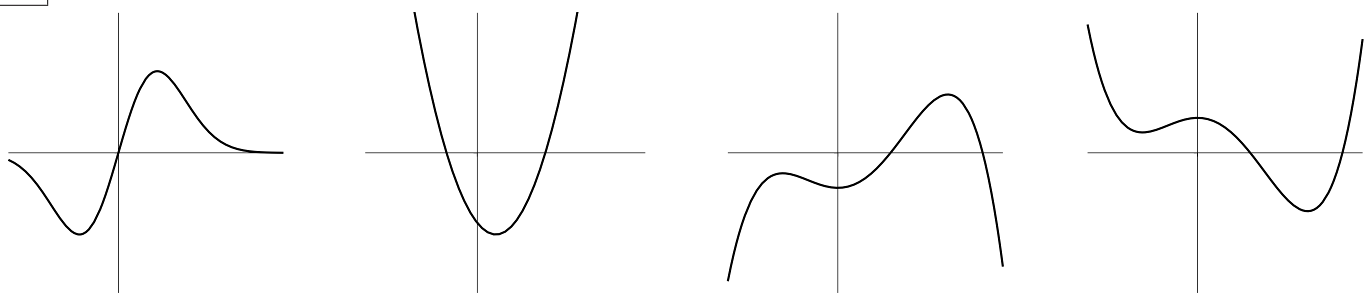
6. At right is the graph of **the derivative** $f'(x)$ of a function $f(x)$. Use it to answer each of the following questions.

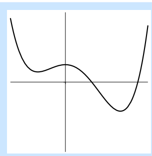


$\boxed{\text{4 points}}$ (a) Is $f(x)$ concave up, concave down, or neither at $x = 0$?

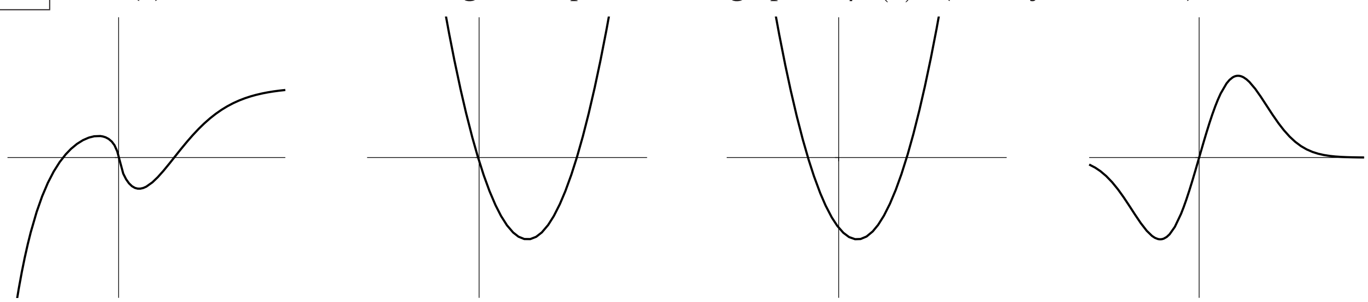
Solution: Since the derivative is decreasing at $x = 0$, we know $f(x)$ is concave down there.

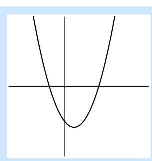
$\boxed{\text{4 points}}$ (b) Which of the following best represents the graph of $f(x)$? (circle your answer).



Solution: The graph of $f(x)$ is 

$\boxed{\text{4 points}}$ (c) Which of the following best represents the graph of $f''(x)$? (circle your answer).



Solution: The graph of $f''(x)$ is 

7. Let $f(x) = \dfrac{x^2 - 3x}{4(x^2 - 9)}$

4 points (a) Identify the horizontal asymptotes of $f(x)$. If there are none, write "NONE".

Solution: A function $f(x)$ has a horizontal asymptote at $y = L$ when $\lim_{x\to\infty} f(x) = L$. So, we have

$$\lim_{x\to\infty} \frac{x^2 - 3x}{4(x^2 - 9)} = \lim_{x\to\infty} \frac{x^2}{4x^2} = \frac{1}{4}.$$

Thus, there is a horizontal asymptote $y = \frac{1}{4}$.

4 points (b) Identify the vertical asymptotes of $f(x)$. If there are none, write "NONE".

Solution: We have a vertical asymptote at $x = a$ whenever $\lim_{x\to a^\pm} f(x) = \pm\infty$. The denominator of $f(x)$ factors as $4(x-3)(x+3)$, so we have to look at $a = 3$ and $a = -3$. Note that if $x \neq 3$ $x \neq -3$, we have

$$f(x) = \frac{x^2 - 3x}{4(x^2 - 9)} = \frac{x(x-3}{4(x-3)(x+3)} = \frac{x}{4(x+3)}$$

Near $x = -3$, we have

$$\lim_{x\to -3^+} f(x) = +\infty \qquad \text{and} \qquad \lim_{x\to -3^-} f(x) = -\infty,$$

so there is a vertical asymptote at $x = -3$.
Near $x = 3$, we have

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x}{4(x+3)} = \frac{3}{4(3+3)} = \frac{1}{8}.$$

Thus, $x = 3$ is not a vertical asymptote.

8 points 8. Write a function which expresses the area of a rectangle with a perimeter of 12 feet in terms of its width.

Solution: Let's let W denote the width of the rectangle (in feet), and L denote its length. Since the perimeter is 12, we know that

$$2L + 2W = 12,$$

or equivalently, $L = 6 - W$.
Since the area of the rectangle is LW, we have

$$A(W) = (6 - W)W$$