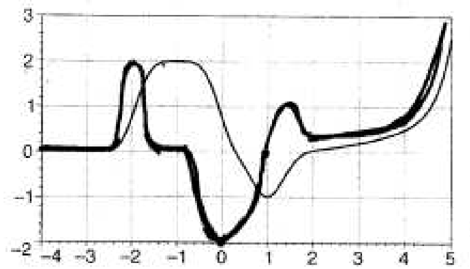
Review for Second Midterm Exam MAT 125

The exam will be held on Tuesday, November 9, at 8:30 PM. Do not forget to bring your student ID card or another photo ID like a driver's license.

1. At right is a graph of a function $f(x)$.
Draw a graph of the derivative $f'(x)$.
At which x values is f increasing?
At which x values is f concave up?



Solution: The derivative is shown sketched on the same graph. Note that:

- f is differentiable everywhere, so f' must be continuous.
- f is essentially flat for $x < -2.5$ and for $-1.5 < x < -\frac{3}{4}$, so $f'(x)$ is nearly zero there.
- for $-2.5 < x < -1.5$, f is increasing and so f' is positive.
- Similarly, f' is positive for $x > 1$ because f is increasing there.
- For $-\frac{3}{4} < x < 1$, $f(x)$ is decreasing, and so f' is negative.
- $f(x)$ is concave up for $-2.5 < x < -2$, so f' is increasing there.
- There is an inflection point at $x = 2$, giving f' a local maximum. Its value is about $f'(x) = 2$, which can be estimated by noticing that f goes up 2 units for every 1 it moves right in that region.
- $f(x)$ is concave down (or flat) between $x = -2$ and $x = 0$, so f' is decreasing (actually, non-increasing) in that range.
- There is an inflection point at $x = 0$ which gives f' a local minimum there.
- Since $f(x)$ is concave up for $0 < x < 1.5$, $f'(x)$ increases to a local maximum at $x = 1.5$.
- $f(x)$ is concave down from about $x = 1.5$ to $x = 2$ or so, and so $f'(x)$ decreases there.
- Finally, $f(x)$ is concave up for $x > 2$, and so $f'(x)$ continues to increase.

2. For each of the following functions $f(x)$, compute the derivative $f'(x)$.

a. $f(x) = (x^3 - x)\ln(x)$

Solution: Using the product rule, we have

$$f'(x) = (3x^2 - 1)\ln(x) + (x^3 - x)\left(\frac{1}{x}\right) = (3x^2 - 1)\ln(x) + x^2 - 1$$

b. $f(x) = e^{2x} - 2x^e$

Solution: $f'(x) = 2e^{2x} - 2ex^{e-1}$

Remember, e is just a number, so the derivative of x^e works just like the derivative of x^3.

c. $f(x) = \left(\dfrac{x^2}{\tan(2x)}\right)^3$

Solution: Use the chain rule and the quotient rule to get

$$f'(x) = 3\left(\frac{x^2}{\tan(2x)}\right)^2 \left(\frac{2x\tan(2x) - 2x^2\sec^2(2x)}{\big(\tan(2x)\big)^2}\right) = \frac{6x^5\left(\tan(2x) - 2x\sec^2(2x)\right)}{\tan^4(2x)}$$

You could use some trig identities to simplify a little more, but it doesn't really help much.

d. $f(x) = \arctan\left(\sqrt{x}\right)$

Solution: This is just the chain rule:

$$f'(x) = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

e. $f(x) = e^{(x^2-\sqrt{x})}$

Solution: Another chain rule problem:

$$f'(x) = e^{(x^2-\sqrt{x})}\left(2x - \frac{1}{2\sqrt{x}}\right)$$

.

f. $f(x) = \sqrt{\dfrac{5}{x} - \dfrac{x}{5}}$

Solution: This is more straightforward if we think of this as $f(x) = \left(5x^{-1} - \frac{1}{5}x\right)^{1/2}$ so that it is just the chain rule, and we don't get confused with the quotient rule. This gives

$$f'(x) = \frac{1}{2}\left(5x^{-1} - \frac{1}{5}x\right)^{-1/2}\left(-5x^{-2} - \frac{1}{5}\right)$$

This can be written as $\dfrac{-25 - x^2}{10x^2\sqrt{5/x - x/5}}$ if you want, or even simplified a little more.

g. $f(x) = \cos(x)\sin(x)$

Solution: Just the use product rule to get

$$f'(x) = (-\sin(x))(\sin(x)) + \cos(x)\cos(x) = \cos^2(x) - \sin^2(x)$$

While this simplifies to $\cos(2x)$ by a standard trig identity, you wouldn't be expected to do that.

3. Let $f(x) = \ln(x)\cos(\pi x)$.

a. Compute $f'(x)$ and find the formula of the tangent line to the graph of $f(x)$ through the point $(1, 0)$.

Solution: Using the product rule and the chain rule, we get

$$f'(x) = -\pi \ln(x)\sin(\pi x) + \frac{1}{x}\cos(\pi x)$$

so $f'(1) = -\pi \cdot 0 \cdot 0 + 1 \cdot (-1) = -1$. This means the tangent line is

$$y = -x + 1$$

.

b. Compute $f''(2)$. Is $f(x)$ concave up or concave down at $x = 2$? Justify your answer.

Solution: Taking the derivative of $f'(x)$ yields

$$f''(x) = -\pi^2 \ln(x)\cos(\pi x) - \frac{\pi}{x}\sin(\pi x) - \frac{1}{x^2}\cos(\pi x) - \frac{1}{x}\pi \sin(\pi x)$$

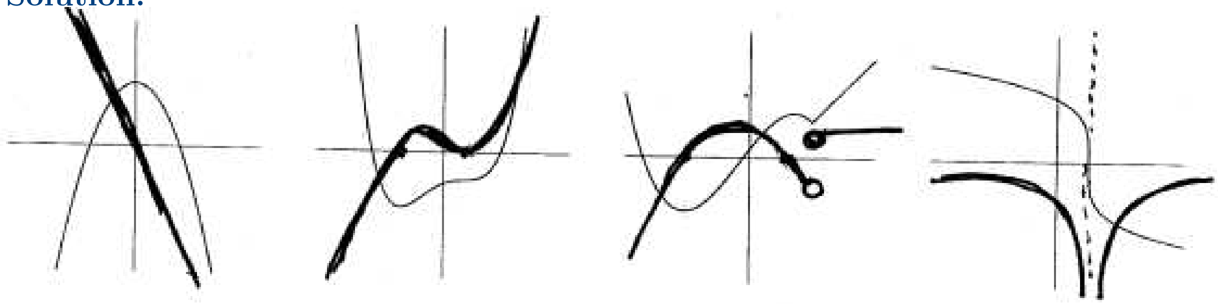
Plugging in $x = 2$ and remembering that $\sin 2\pi = 0$ and $\cos 2\pi = 1$ gives

$$f''(2) = -\pi^2 \ln(2) - \frac{1}{4}$$

Since $f''(2) < 0$, we know $f(x)$ is concave downward at $x = 2$.

4. The graphs of several functions $f(x)$ are shown below. On the same set of axes, sketch the function $f'(x)$.

Solution:



There isn't a lot to say here. The only slightly tricky bits are to notice that the third graph has a point where $f(x)$ is not differentiable, so $f'(x)$ doesn't exist there (and has a discontinuity, going from negative to positive. The fourth graph is constantly decreasing, with a vertical tangent, meaning $f'(x) < 0$ and $f'(x)$ has an asymptote there.

5. Let $f(x) = 2x^2 + xe^x$.

a. Compute $f'(x)$ and $f''(x)$.

Solution:

$$f'(x) = 4x + e^x + xe^x$$
$$f''(x) = 4 + e^x + e^x xe^x = 4 + 2e^x + xe^x$$

b. For which of the following values of x, is $f(x)$ increasing near x? -1, 0, 2.

Solution: $f'(-1) = -4 + e^{-1} - e^{-1} = -4 < 0$, so f is not increasing at $x = -1$.

$f'(0) = 0 + e^0 + 0 = 1 > 0$, so f is increasing at $x = 0$.

$f'(2) = 8 + e^2 + 2e^2 = 8 + 3e^2 > 0$, so f is increasing at $x = 2$.

c. Is $f(x)$ concave up near $x = 0$?

Solution: Since $f''(0) = 4 + 2 = 6 > 0$, yes it is.

d. Are there any values of x for which $f(x)$ is concave down near x?

(**Hint:** Remember that $e^t > 0$ for all values of t).

Solution: No, since $f''(x) = 4 + (2 + x)e^x$ is always positive. In order to have $f''(x) < 0$, certainly $x < -2$, but e^x is so small for $x < -2$ that the product can never be less than -4.

6. Do the following limits exist? If they do, compute them:

a. $\displaystyle\lim_{x\to\infty} \frac{7x^4 + 4x}{5x^5 - 6x + 1}$

Solution: Since x is getting large, the highest powers dominate, so this limit can be done as

$$\lim_{x\to\infty} \frac{7x^4}{5x^5} = \lim_{x\to\infty} \frac{7}{5x} = 0$$

b. $\displaystyle\lim_{x\to-\infty} \frac{|x|}{x}$

Solution: For $x < 0$, $|x| = -x$ so we do this as

$$\lim_{x\to-\infty} \frac{-x}{x} = \lim_{x\to-\infty} -1 = -1$$

c. $\displaystyle\lim_{x\to 2} \frac{1}{2-x}$

Solution: This limit does not exist, since if $x < 2$, $\frac{1}{2-x} > 0$, while for $x > 2$, $\frac{1}{2-x} < 0$.

Note that $\displaystyle\lim_{x\to 2^+} \frac{1}{2-x} = -\infty$ and $\displaystyle\lim_{x\to 2^-} \frac{1}{2-x} = +\infty$.

d. $\displaystyle\lim_{x\to 2} \frac{1}{2-x}$

Solution: Oops. It seems I wrote this one twice. See above.

e. $\displaystyle\lim_{x\to\infty} \frac{\sin(x^2)}{x}$

(**Hint:** Remember that $-1 \le \sin(t) \le 1$ for all values of t).

Solution: Using the squeeze theorem, we have

$$0 = \lim_{x\to\infty} \frac{-1}{x} \le \lim_{x\to\infty} \frac{\sin(x^2)}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0$$

So the limit is 0.

f. $\displaystyle\lim_{x\to 3^+} \ln(x^2 - 4x + 3)$

Solution: As $x \to 3^+$, $x^2 - 4x + 3 \to 0^+$, so this limit is the same as $\displaystyle\lim_{y\to 0^+} \ln(y)$. This limit is $-\infty$.

7. At right is the graph of a function $f(x)$. Do the following limits exist? If they do, what are they?

a. $\displaystyle\lim_{x\to -3} f(x)$

Solution: $+\infty$

b. $\displaystyle\lim_{x\to 0^+} f(x)$

Solution: $+\infty$

c. $\displaystyle\lim_{x\to 3^-} f(x)$

Solution: $-\infty$

d. $\displaystyle\lim_{x\to 3} f(x)$

Solution: The limit does not exist.

e. $\displaystyle\lim_{x\to -\infty} f(x)$

Solution: 2

f. $\displaystyle\lim_{x\to\infty} f(x)$

Solution: Does not exist, because $f(x)$ keeps wiggling.

g. $\displaystyle\lim_{x\to\infty} \frac{f(x)}{x}$

Solution: This limit is 0 by the squeeze theorem:

$$0 \le \lim_{x\to\infty} \frac{f(x)}{x} \le \lim_{x\to\infty} \frac{6}{x} = 0$$

8. Which of the following represents $f'(2)$ where $f(x) = e^{x^2}$.

$$\lim_{x\to 2} \frac{e^{x^2} - e^{a^2}}{h} \qquad \lim_{h\to 0} \frac{e^4(e^{4h+h^2} - 1)}{h} \qquad \lim_{x\to 2} \frac{e^{x^2} - e^2}{x-2} \qquad \lim_{h\to 0} \frac{e^{(x^2+h^2)} - e^{x^2}}{h}$$

Solution: By the definition of the limit,

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{e^{(2+h)^2} - e^4}{h} = \lim_{h\to 0} \frac{e^{4+4h+h^2} - e^4}{h} = \lim_{h\to 0} \frac{e^4 e^{4h+h^2} - e^4}{h}$$

$$= \lim_{h\to 0} \frac{e^4\left(e^{4h+h^2} - 1\right)}{h}$$

Each of the other choices has some problem with it.

9. If $h(0) = 1$, $h'(0) = 3$, $h(1) = 0$, $h'(1) = -1$, $h(2) = 0$, $h'(2) = 2$, and $f(x) = h(h(x))$, what is $f'(1)$?

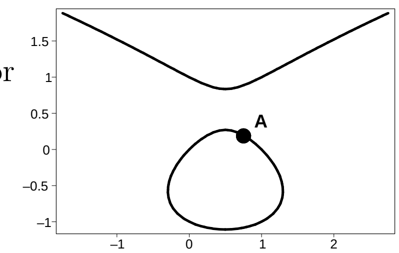
Solution:

$$f'(1) = h'(h(1)) \cdot h'(1) = h'(0) \cdot (-1) = 3 \cdot (-1) = -3.$$

10. Consider the curve C which consists of the set of points for which

$$x^2 - x = y^3 - y$$

(see the graph at right).



a. Write the equation of the line tangent to C at the point $(1, 0)$.

Solution: We use implicit differentiation to get $2x - 1 = 3y^2y' - y'$. Plugging in $x = 1$ and $y = 0$ gives $2 - 1 = 0 - y'$, so

$$y' = -1 \qquad \text{at } (1, 0).$$

This means the tangent line is

$$y = -x + 1$$

b. Use your answer to part **a** to estimate the y-coordinate of the point with x-coordinate 3/4 marked A in the figure. Plug your estimate into the equation for C to determine how good it is.

Solution: When $x = 3/4$, we have $y = -3/4 + 1$ on the tangent line, so the point $(3/4, 1/4)$ should be close to the curve C.

If we plug in $x = \frac{3}{4}$ and $y = \frac{1}{4}$ to the original equation, we get $\dfrac{9}{16} - \dfrac{3}{4} \approx \dfrac{1}{64} - \dfrac{1}{4}$. If we write this all with a denominator of 64, we get

$$\frac{-12}{64} \approx \frac{-15}{64}$$

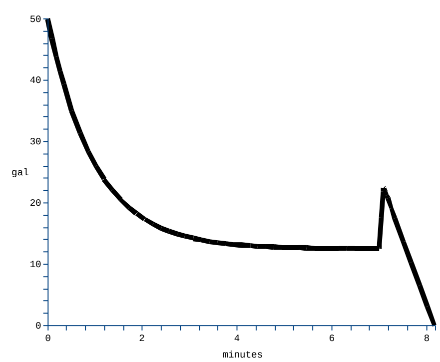
so we have an error of about $3/64$ or 0.047.

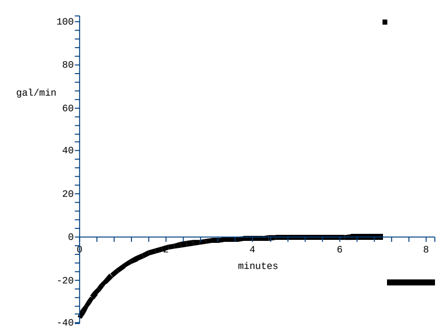
11. In the paragraph below is a description of how the amount of water $W(t)$ in a tub varied with time.

> The tub held about 50 gallons of green, brackish water, with some stuff floating in it that I didn't even want to guess about. I had to get it out of there. When I opened the drain the water drained out rapidly at first, but then it went slower and slower, until it stopped completely after about 5 minutes. The tub was about 1/4-full of that nasty stuff. Would I have to stick my hand in it? *Ick*— there was no way I could do that. I just stared at it for a couple of minutes, but then I got an idea. I dumped in about 10 gallons of boiling water. That did something: there was this tremendous noise like *BLUUUUURP*, and then the tub drained steadily, emptying completely in just a minute or so.

Use this description to sketch a graph of $W(t)$ and its derivative $W'(t)$. Pay careful attention to slope and concavity. Label the axes, with units.

Solution: A pair of graphs something like those below agrees with the description (the graph of $W(t)$ is on the left, its derivative on the right). The graph starts out at 50, then decreases "slower and slower", (which is another way of saying it is decreasing and concave up) until it finally flattens out at about 5 minutes with a value of $12\frac{1}{2}$. The "spike" at around 7 minutes corresponds to when the 10 gallons of boiling water were added, raising the amount to $22\frac{1}{2}$, and then the level drops with constant slope, hitting the axis just about a minute later.





Of course, you might have minor variations. For example, the region around 7 minutes could be smooth, or discontinuous.