Expected Solutions

First Midterm Exam MAT 125, Spring 1999

problem	1	2	3	4	5	6	7	Total
possible	10	10	20	18	12	15	15	100
score								

Name: **Section:**

Directions: There are 7 problems on five pages in this exam. Make sure that you have them all. Do all of your work in this exam booklet, and cross out any work that the grader should ignore. You may use the backs of pages, but indicate what is where if you expect someone to look at it. Books, extra papers, and discussions with friends are not permitted. Leave all your exact form (that is, do *not* approximate π, square roots, and so on.)

1. *(10 points)* Let

$$f(x) = \begin{cases} x^3 & x < 0 \\ \tan x & 0 \le x < \frac{\pi}{4} \\ 1 & x \ge \frac{\pi}{4} \end{cases}$$

For which values of x is $f(x)$ continuous? Justify your answer.

The graph is continuous for all values of x.

For $x<0$, $f(x) = x^3$ is a polynomial which is continuous everywhere.

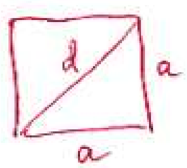
For $x=0$ $f(x) = \tan(0) = 0$ and $\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} x^3 = 0$, $\lim_{x\to 0^+} = \lim_{x\to 0^+} \tan x = 0$.

For $0<x<\frac{\pi}{4}$, $f(x) = \tan(x)$ is continuous since $\tan(x)$ is trigonometric and hence continuous in its domain, which is $\{x \neq (n+\frac{1}{2})\pi$, n an integer$\}$

for $x=\frac{\pi}{4}$, $\lim_{x\to\frac{\pi}{4}^+} f(x) = \lim_{x\to\frac{\pi}{4}^+} 1 = 1$ and $\lim_{x\to\frac{\pi}{4}^-} f(x) = \lim_{x\to\frac{\pi}{4}^-} \tan(x) = 1 = f(\frac{\pi}{4})$.

For $x>\frac{\pi}{4}$ $f(x)=1$ is a constant and therefore continuous.

2. *(10 points)* Let Q be a square, and let d be the length of its diagonal. Write a function which expresses the area of Q as a function of d, and state the domain of this function.



$$\boxed{\text{Area} = a^2 = \frac{d^2}{2}}$$

$a^2 + a^2 = d^2$

$2a^2 = d^2$

$a^2 = \frac{d^2}{2}$

$$\boxed{\text{Domain} = \{d ; d>0\}}$$

3. *(20 points)* Compute each of the limits below. If the limit does not exist, say so. Justify your answer in all cases.

a. $\lim_{x\to 2} xe^{x-2}$

The function xe^{x-2} is continuous, therefore the $\lim_{x\to 2}$ = value of the function at 2

So: $\lim_{x\to 2} xe^{x-2} = 2\cdot e^{2-2} = 2\cdot e^0 = 2\cdot 1 = 2$

b. $\lim_{x\to 3} \dfrac{x^2-4x+12}{x-3}$

$x=3$ is not in the domain of this function, and if you plug in $x=3$ you get $\frac{9}{0}$, so you cannot factor $(x-3)$ from the numerator and simplify by it.

Therefore the limit DOES NOT exist.

c. $\lim_{x\to 0+} x\cos(\ln x)$ Hint: recall that $-1 \le \cos x \le 1$ for any x.

Use the hint, it suggest for the Squeeze Thm.

$-1 \le \cos x \le 1$ so $-1 \le \cos(\ln x) \le 1$ and $-x \le x\cos(\ln x) \le x$

Therefore: $\underbrace{\lim_{x\to 0+}(-x)}_{0} \le \lim_{x\to 0+} x\cos(\ln x) \le \underbrace{\lim_{x\to 0+} x}_{0}$ you get $\lim_{x\to 0+} x\cos(\ln x) = 0$

d. $\lim_{x\to 0} \dfrac{(2-x)^2-4}{x}$

If you plug in $x=0$ you get $\frac{0}{0}$ so you are able to simplify.

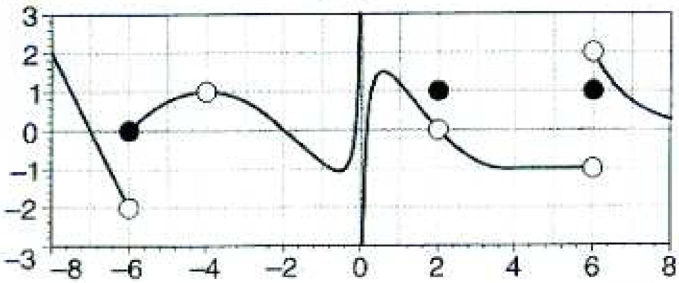
$\lim_{x\to 0}\dfrac{(2-x)^2-4}{x} = \lim_{x\to 0}\dfrac{4-4x+x^2-4}{x} = \lim_{x\to 0}\dfrac{x^2-4x}{x} = \lim_{x\to 0}\dfrac{x(x-4)}{x} =$

$= \lim_{x\to 0}(x-4) = -4$

e. $\lim_{x\to\pi}\left(\sin x + e^{-\sqrt[7]{(x-\pi)^2}}\right)$ This is a continuous function, therefore the limit as $x\to\pi$ is the same as the value of the function at $x=\pi$.

So $\lim_{x\to\pi}\left(\sin x + e^{-\sqrt[7]{(x-\pi)^2}}\right) = \sin\pi + e^{-\sqrt[7]{(\pi-\pi)^2}} = \sin\pi + e^{-\sqrt[7]{0}} = 0 + e^0 = 1$

4. *(18 points)* Let $f(x)$ be the function whose graph is shown below.



a. Circle each value of x listed below where $f(x)$ is continuous from the right.

-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7

b. List all points $-8 < x < 8$ where $f(x)$ is not continuous. If there are none, write "none".

$x = -6, -4, 0, 2, 6$

c. What is $f(6)$? If it is not defined, write DNE.

$f(6) = 1$

d. What is $\lim_{x \to 2} f(x)$? If it is not defined, write DNE.

$\lim_{x \to 2} f(x) = 0$

e. What is $\lim_{x \to 6^+} f(x)$? If it is not defined, write DNE.

$\lim_{x \to 6^+} f(x) = 2$

f. What is $\lim_{x \to 4} (f(x/2) - f(-x)/2)$? If it is not defined, write DNE.

$\lim_{x \to 4} \left(f(x/2) - f(-x)/2 \right) = \lim_{x \to 2} f(x) - \frac{\lim_{x \to 4} f(-x)}{2} = 0 - \frac{1}{2} = -\frac{1}{2}$

5. *(12 points)* Let $f(x) = \sqrt{\frac{x-1}{x}}$.

a. What is the domain of $f(x)$?

We want $\frac{x-1}{x} \geq 0$. So $x-1 \geq 0$ and, therefore $x \geq 1$ OR $x < 0$. $\therefore \{x \geq 1\} \cup \{x < 0\}$.

b. Find two functions g and h such that $f = g \circ h$.

Let $h(x) = \frac{x-1}{x}$ and $g(x) = \sqrt{x}$ [other answers possible]

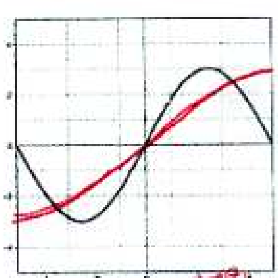
c. Find a formula for $f^{-1}(x)$.

let $y = \sqrt{\frac{x-1}{x}}$ $\therefore$ $y^2 = \frac{x-1}{x} \Rightarrow xy^2 = x - 1$

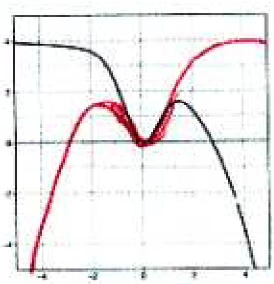
$\Rightarrow xy^2 - x = -1$ & $x(y^2 - 1) = -1$

$\therefore x = \frac{-1}{y^2-1}$ $\therefore f^{-1}(x) = \frac{-1}{x^2-1}$

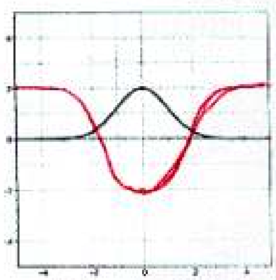
6. *(15 points)* The graphs of several functions $f(x)$ are shown below. On the same set of axes, sketch the function $g(x)$ as indicated.



$g(x) = f(x/2)$ HORIZ. COMPRESS BY 1/2 (IE STRETCH BY 2)



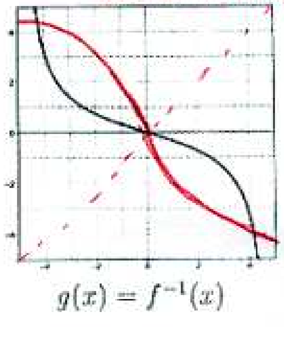
$g(x) = f(-x)$



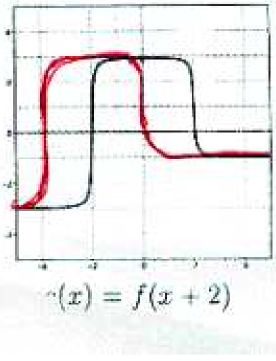
$g(x) = 2 - 2f(x)$

FLIP THRU X AXIS SCALE BY 2 VERTICALLY THEN MOVE UP 2 UNITS

MIRROR THROUGH $y = x$ LINE.



$g(x) = f^{-1}(x)$



$g(x) = f(x+2)$

SHIFT HORIZ. 2 UNITS LEFT.

7. *(15 points)*

a. The values of the functions h and g are given by the table at right. What is the value of the function $h \circ g$ at 1?

x	g(x)	h(x)
0	1	2
1	0	1
2	2	0

$h(g(1)) = h(0) = 2$.

b. If $5e^{2x} = 10$, what is x?

$e^{2x} = 2$

$\ln(e^{2x}) = \ln 2$

$2x \ln(e) = \ln 2$

SINCE $\ln(e) = 1$

$2x = \ln 2$

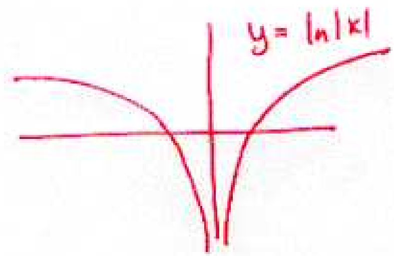
$x = \frac{1}{2}\ln 2$

c. What is the domain of the function $\ln|x|$?

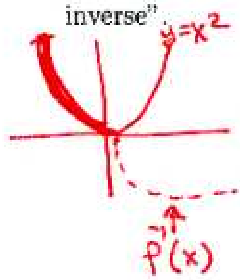
THE DOMAIN OF $\ln(x)$ IS $x > 0$, SO WE NEED $|x| > 0$.

THIS HAPPENS WHENEVER $x \neq 0$

SO THE DOMAIN OF $\ln|x|$ IS ALL $x \neq 0$



d. What is the inverse of $f(x) = x^2$, with $x < 0$? If the function is not invertible, write "no inverse".



IF WE RESTRICT OUR ATTENTION TO $x<0$, WE HAVE $y = x^2$ $(x<0)$

SO $\pm\sqrt{y} = x$. SINCE $x < 0$, WE WANT THE NEGATIVE SIGN.

SO $f^{-1}(x) = -\sqrt{x}$

e. Write the equation of the line passing through the point $(-2, 4)$ that has slope 3.

$y - 4 = 3(x+2)$

$y = 3x + 6 + 4$

$\boxed{y = 3x + 10}$