MAT 123 Midterm Exam Solutions

Question 1: Consider the function

$$f(x) = \frac{x^2 - 2x + 1}{3x^2 - 27}$$

(a) *(3 pts)* What is the domain of this function? Write your answer using interval notation.

As this is a rational function, the domain will be all real numbers except those that make the denominator zero. The denominator factors as

$$3x^2 - 27 = 3(x^2 - 9) = 3(x - 3)(x + 3)$$

so the denominator will equal zero for $x = \pm 3$. So the domain of the function is all real numbers except ± 3, which, in interval notation, is

$$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$$

(b) *(3 pts)* What are the zeros of this function?

The zeroes of a rational function are the same as the zeroes of the numerator, provided that they are in the domain. The numerator factors as

$$x^2 - 2x + 1 = (x - 1)^2$$

so the only zero of the numerator is $x = 1$ (which is a repeated root). Note that this is in the domain.

(c) *(4 pts)* Describe the end behavior of this function.

The end behavior is determined by the highest degree terms on top and bottom, so as x gets very large or very small, the values of $f(x)$ will be close to the values of

$$\frac{x^2}{3x^2} = \frac{1}{3}$$

So the end behavior is

$$\lim_{x\to\infty} f(x) = \frac{1}{3} \qquad \lim_{x\to-\infty} f(x) = \frac{1}{3}$$

(Note that this means the function will have a horizontal asymptote at $y = \frac{1}{3}$).

(d) *(1 pt)* Find the y-intercept of the graph.

Recall that the y-intercept is given by the function's value at $x = 0$, so we compute

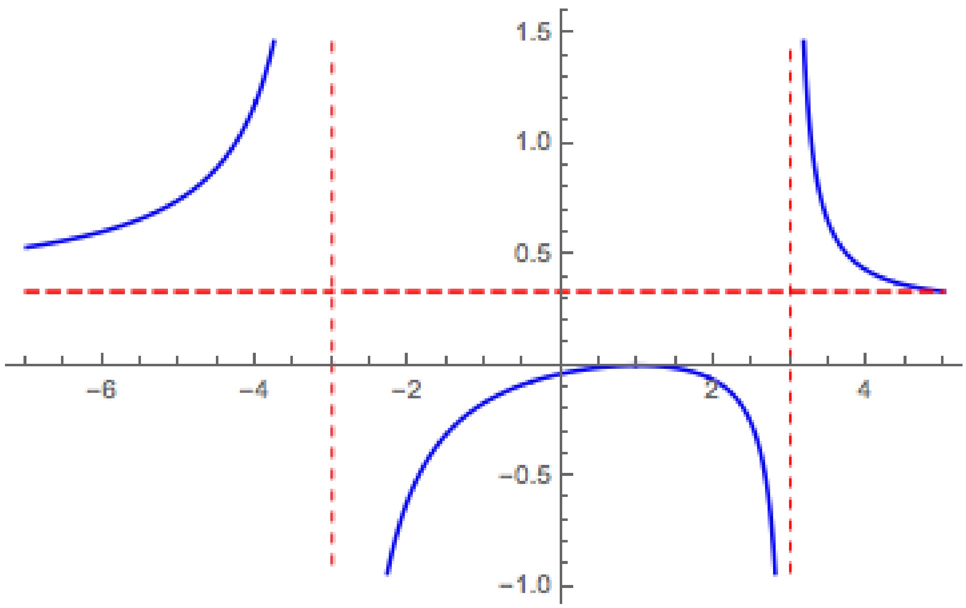
$$f(0) = \frac{0^2 - 2 \cdot 0 + 1}{3 \cdot 0^2 - 27} = \frac{1}{-27}$$

So the y-intercept is at $\frac{-1}{27}$.

(e) *(5 pts)* Using the information from the previous parts of this problem, sketch a graph of this function. Your graph should include all asymptotes and the x and y intercepts. Some values of the function are computed below to help you graph the function -

$$f(-4) = \frac{25}{21} \qquad f(-2) = \frac{-3}{5} \qquad f(2) = \frac{-1}{15} \qquad f(4) = \frac{3}{7}$$

Parts (a) and (b) together tell us that the graph will have vertical asymptotes at $x = -3$ and $x = 3$. Part (c) tells us that the graph will have a horizontal asymptote at $y = \frac{1}{3}$. We then can plot the x- and $y-$intercepts, and finally use the points given above to help us draw each part of the graph. The graph is shown below - the function is graphed in blue, and the asymptotes are graphed in dotted red lines.



Question 2: Consider the points $A = (-1, -4), B = (7, -1)$ and $C = (1, -10)$ in the xy-plane.

(a) *(3 pts)* Write an equation for the line that passes through the points B and C.

Using the slope formula, the line will have slope

$$m = \frac{-10 - (-1)}{1 - 7} = \frac{-9}{-6} = \frac{3}{2}$$

Any of the following answers would then be acceptable. The first two are the point-slope form, using the points B and C respectively, and the third is the same line written in slope-intercept form.

$$\begin{aligned} y + 1 &= \frac{3}{2}(x - 7) \\ y + 10 &= \frac{3}{2}(x - 1) \\ y &= \frac{3}{2}x - \frac{23}{2} \end{aligned}$$

(b) *(2 pts)* Write an equation for the line that is parallel to the line from part (a) and passes through the point A.

Since the line we are trying to write is parallel to the line from part (a), it will have the same slope, $m = \frac{3}{2}$. Using point-slope form with this slope and the point A gives the equation.

$$y + 4 = \frac{3}{2}(x + 1)$$

In slope-intercept form the equation is

$$y = \frac{3}{2}x - \frac{5}{2}$$

(c) *(2 pts)* Write an equation for the line that is perpendicular to the line from part (a) and passes through the point A.

Since the line we are trying to write is perpendicular to the line from part (a), it will have slope, $m = \frac{-2}{3}$. Using point-slope form with this slope and the point A gives the equation.

$$y + 4 = \frac{-2}{3}(x + 1)$$

In slope-intercept form the equation is

$$y = \frac{-2}{3}x - \frac{14}{3}$$

Question 3: Let $f(x) = 3^x$ and let $g(x) = 2x + 1$.

(a) *(3 pts)* Write a formula for $h(x) = (g \circ f)(x)$. Using that formula, describe in words how the graph of $h(x)$ can be obtained from the graph of $f(x)$.

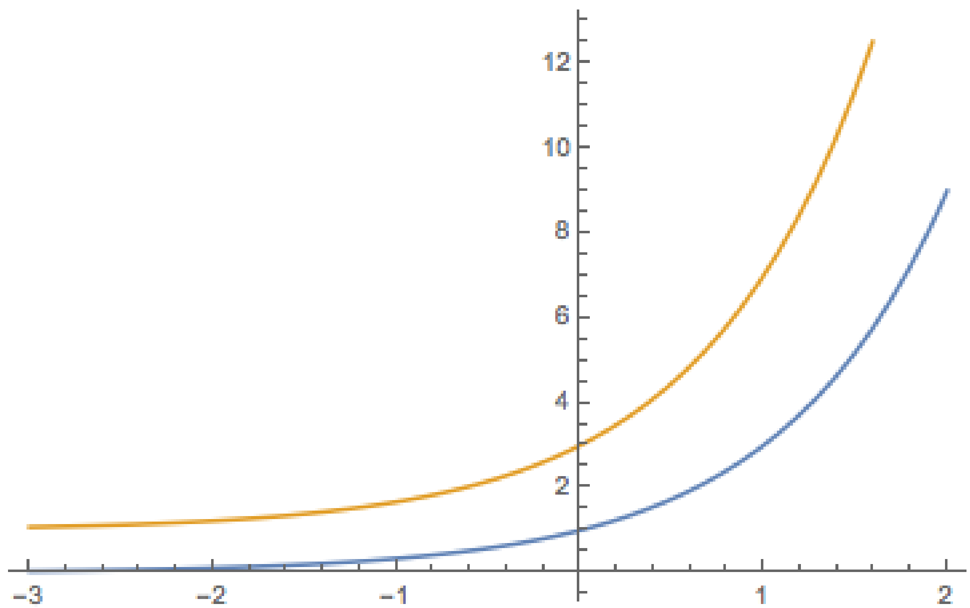
$$\begin{aligned} h(x) &= (g \circ f)(x) = g(f(x)) \\ &= g(3^x) \\ &= 2 \cdot 3^x + 1 \end{aligned}$$

Interpreting this using graph transformations, the graph of $h(x)$ is obtained from the graph of $f(x)$ by a vertical stretch by 2, followed by a shift up one unit.

(b) *(4 pts)* Draw the graph of $f(x) = 3^x$. Label at least two points on the graph. On the same set of axes, draw the graph of $h(x)$, and label at least two points on the graph.

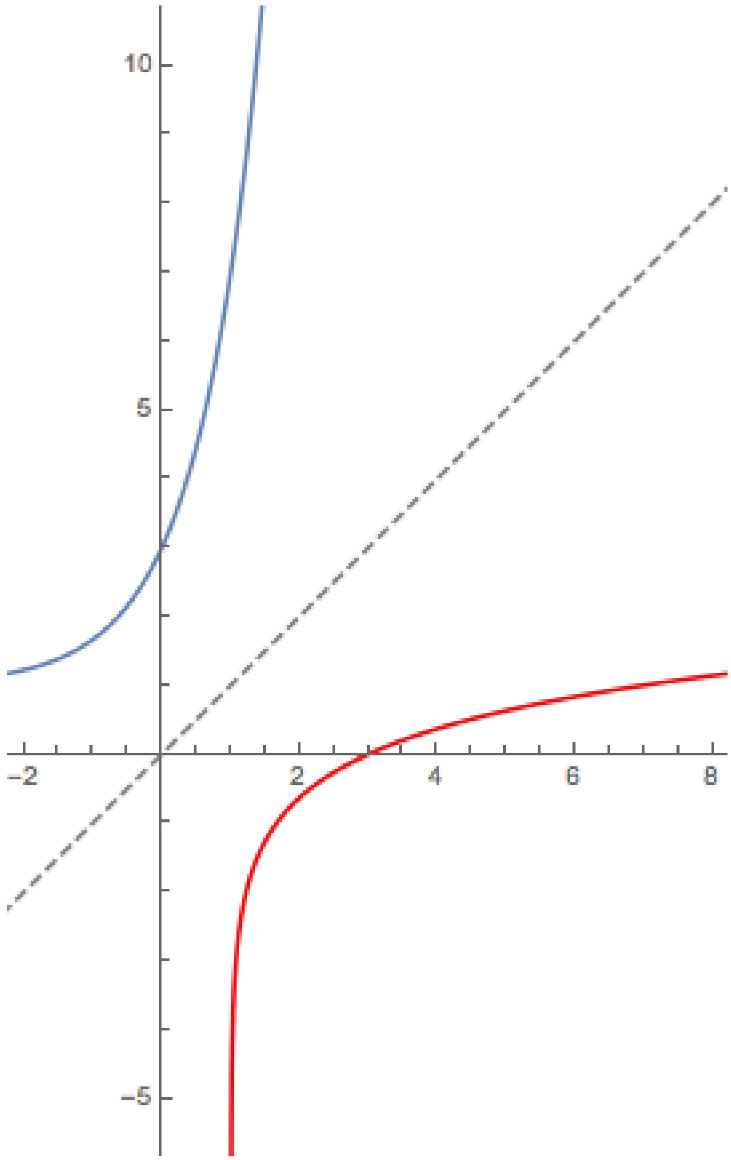
The graph of $f(x) = 3^x$ has the standard shape of an exponential graph. It is show in blue below. Notice that the graph has a horizontal asymptote on the left at $y = 0$, and that the graph passes through the points $(0, 1)$ and $(1, 3)$.

The graph of $h(x)$ is shown in gold. Notice that it has a horizontal asymptote on the left at $y = 1$, which is shifted up 1 unit from the asymptote of the graph of f. It passes through the points $(0, 3)$ and $(1, 7)$.



(c) *(3 pts)* If you've drawn the graph of $h(x)$ correctly, you should notices that $h(x)$ is invertible. Draw the graph of $h^{-1}(x)$. Label at least two points on the graph, and make sure your graph has the proper domain and range.

Recall that the graph of the inverse function is obtained by reflecting the graph of the original function over the line $y = x$. As the original function has a horizontal asymptote at $y = 1$, the inverse function will have a vertical asymptote at $x = 1$. The original function passes through the points $(0,3)$ and $(1,7)$, so the new function will pass through the points $(3,0)$ and $(7,1)$. The graphs of both $h(x)$ and $h^{-1}(x)$ are shown below. The original graph $h(x)$ is in blue, the inverse $h^{-1}(x)$ is in red, and the line $y=x$ is shown for clarity.



(d) *(3 pts)* Find a formula for $h^{-1}(x)$. (Your formula should involve a logarithm.)

Writing $y = 2\cdot 3^x + 1$, we solve that equation for x to find the inverse function. This gives

$$\begin{aligned} y &= 2\cdot 3^x + 1 \\ y-1 &= 2\cdot 3^x \\ \frac{y-1}{2} &= 3^x \\ \log_3\left(\frac{y-1}{2}\right) &= x \end{aligned}$$

So the inverse function is

$$f^{-1}(y) = \log_3\left(\frac{y-1}{2}\right)$$

Question 4: Let $M(t) = -2t^2 - 8t - 13$.

(a) *(3 pts)* Write the equation for this quadratic in vertex form.

We'll use the shortcut method to write this in vertex form, instead of completing the square. Using that $h = \frac{-b}{2a}$, we have

$$h = \frac{-b}{2a} = \frac{-(-8)}{2(-2)} = \frac{8}{-4} = -2$$

The value of k is then $M(h)$, which we find is

$$k = M(h) = -2 \cdot (-2)^2 - 8(-2) - 13 = -8 + 16 - 13 = -5$$

So in vertex form, the equation is

$$M(t) = -2(t + 2)^2 - 5$$

(b) *(2 pts)* Does this function have a maximum value? If so, what is it? Does this function have a minimum value? If so, what is it?

The minimum or maximum value of a quadratic function always happens at the vertex. If a is positive, the parabola opens up and the vertex is the minimum value. If a is negative, the parabola opens down and the vertex is the maximum value. In this case, we have $a = -2$, so the parabola opens up and we have a maximum, but no minimum. The maximum value is then the y-coordinate of the vertex, which is -5.

Question 5: Solve the following equations, or explain why there is no solution.

(a) *(3 pts)* $\log_3\left(\frac{1}{9}\right) = \log_x 25$

On the left hand side of this equation, we observe that

$$\log_3\left(\frac{1}{9}\right) = -2$$

since

$$3^{-2} = \frac{1}{9}$$

So the equation can be rewritten as

$$-2 = \log_x 25$$

Rewriting this as an exponential equation, we want to find x such that

$$x^{-2} = 25$$

Taking both sides to the $-1/2$ power gives

$$x = 25^{-1/2} = \frac{1}{\sqrt{25}} = \frac{1}{5}$$

So the solution is $x = \frac{1}{5}$.

(b) *(3 pts)* $2x^2 - 4x - 10 = 20$

First, we rearrange the equation to

$$2x^2 - 4x - 30 = 0$$

We then notice that we can factor out a 2, yielding

$$2(x^2 - 2x - 15) = 0$$

Finally, we can factor the quadratic as

$$2(x-5)(x+3) = 0$$

Setting each factor to zero gives the solutions $x = 5, -3$.

(c) *(3 pts)* $3x^2 + 5x + 3 = 0$

This equation does not factor, so we have to use the quadratic formula. The solutions are then

$$\begin{aligned} x &= \frac{-5 \pm \sqrt{5^2 - 4 \cdot 3 \cdot 3}}{2 \cdot 3} \\ &= \frac{-5 \pm \sqrt{-11}}{6} \end{aligned}$$

We can't take the square root of a negative number, so this equation has no solutions.

Question 6: Consider the polynomial $p(x)$ defined by

$$p(x) = \frac{-1}{2}(x^2 - 4)(x^2 + 1)$$

(a) *(3 pts)* Find the zeros of this polynomial.

Recall that we find the zeroes of a polynomial by factoring and setting each factor equal to 0. Luckily this polynomial is already factored, so setting each factor equal to zero gives the equations $x^2 - 4 = 0$ and $x^2 + 1 = 0$. Solving the first, we have

$$\begin{aligned} x^2 - 4 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Attempting to solve the second gives

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \end{aligned}$$

No real number squares to a negative number, so this has no solutions. Thus the only zeroes for our polynomial are $x = \pm 2$.

(b) *(3 pts)* Identify the degree and leading coefficient of $p(x)$.

To find the degree and leading coefficient, we need to multiply out the factors on the right hand side to write the polynomial in standard form. This gives

$$\begin{aligned} p(x) &= \frac{-1}{2}(x^4 - 3x^2 - 4) \\ &= \frac{-1}{2}x^4 + \frac{3}{2}x^2 + 2 \end{aligned}$$

The degree is 4, and the leading coefficient it $\frac{-1}{2}$.

(c) *(3 pts)* Describe the end behavior of this polynomial.

Recall that we only look at the highest degree term to determine the end behavior. If we plug a very large number into the highest degree term $\frac{-1}{2}x^4$, taking that number to the fourth power will give a very large positive number, which will give a large negative number when multiplied by the negative leading coefficient.. Thus we have

$$\lim_{x \to \infty} p(x) = -\infty$$

If we plug a very large negative number into $\frac{-1}{2}x^4$, taking that number to an even power will give a very large positive number, which will give a large negative number when multiplied by the negative leading coefficient.. Thus we have

$$\lim_{x \to -\infty} p(x) = -\infty$$

(d) *(2 pts)* Find the values of $p(-1)$ and $p(1)$.

Computing these values gives

$$\begin{aligned} p(-1) &= \frac{-1}{2}((-1)^2 - 4)((-1)^2 + 1) \\ &= \frac{-1}{2}(-3)(2) \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} p(-1) &= \frac{-1}{2}((1)^2 - 4)((1)^2 + 1) \\ &= \frac{-1}{2}(-3)(2) \\ &= 3 \end{aligned}$$

(e) *(4 pts)* Plot a graph of the polynomial. Your graph should have the proper x and y intercepts, end behavior, and number of "turns," and should pass through the points you found in part (d).

We have already figured out the x-intercepts (at $x = \pm 2$) and the end behavior, plus the points from part (d). The y-intercept is the value of the function at zero, which we can compute is $p(0) = 2$. Finally, our polynomial is degree 4, so should have no more than three "turns." Putting this together in a graph gives