MAT 123 Practice for Midterm 2 with Solutions

Remark. If you are comfortable with all of the following problems, you will be well prepared for Midterm 1.

Exam Policies. You must show up on time for all exams. Please bring your student ID card: ID cards may be checked, and students may be asked to sign a picture sheet when turning in exams. Other policies for exams will be announced / repeated at the beginning of the exam.

If you have a university-approved reason for taking an exam at a time different than the scheduled exam (because of a religious observance, a student-athlete event, etc.), please contact your instructor as soon as possible. Similarly, if you have a documented medical emergency which prevents you from showing up for an exam, again contact your instructor as soon as possible.

All exams are closed notes and closed book. Once the exam has begun, having notes or books on the desk or in view will be considered cheating and will be referred to the Academic Judiciary.

It is not permitted to use cell phones, calculators, laptops, MP3 players, Blackberries or other such electronic devices at any time during exams. If you use a hearing aid or other such device, you should make your instructor aware of this before the exam begins. You must turn off your cell phone, etc., prior to the beginning of the exam. If you need to leave the exam room for any reason before the end of the exam, it is still not permitted to use such devices. Once the exam has begun, use of such devices or having such devices in view will be considered cheating and will be referred to the Academic Judiciary. Similarly, once the exam has begun any communication with a person other than the instructor or proctor will be considered cheating and will be referred to the Academic Judiciary.

Mastery Review Topics.
In addition to the Core Competency Exam, each midterm and final will include mastery questions. Students who have already passed the Core Competency Exam need not repeat the Core Competency Exam: those students may devote all of their time and energy to the mastery questions.
For Midterm 2, these are the mastery topics.

(1) For a given equation of an ellipse, find the center of the ellipse, and find the horizontal and vertical distances from the center to points on the ellipse (the semimajor / semiminor axes).

(2) Find points on a given conic section satisfying specified conditions, e.g., lying on a given line.

(3) Solve problems that are applications of the quadratic formula, i.e., find a pair of real numbers with given sum and product, solve problems involving distances in the plane, etc.

(4) For power functions and their variations, find the domain and range, say whether the function is invertible, and find an inverse when it exists.

(5) For a higher-degree polynomial, when the zeroes are given or are easy to find using only the quadratic formula, factoring, etc., give a rough sketch of the function illustrating the x-intercepts, the y-intercepts, and behavior at $\pm\infty$.

(6) Perform algebraic manipulations with polynomial functions.

(7) For exponential and logarithm functions and their variations, find the domain and range, say whether the function is invertible, and find an inverse when it exists.

(8) Solve problems involving exponential growth / decy models, e.g., population growth and radioactive decay. Understand the doubling time for population growth, resp. half-life of radioactive decay.

(9) Solve problems involving logarithmic growth, e.g., number of digit problems, problems involving the Richter scale, and problems involving the Decibel scale.

Practice Problems.

(1) In each of the following cases, find the center of the given ellipse. Find the distances from the center to the point on the ellipse contained in the unique horizontal / vertical line containing the center.

$$(a)\ 4x^2 + 8x + y^2 - 2y = 11, \quad (b)\ x^2 + 2x + 4y^2 + 24y = -36,$$
$$(c)\ 9x^2 + 36x + y^2 - 10y + 60 = 0, \quad (d)\ 9x^2 - 54x + 4y^2 + 8y + 49 = 0.$$

Solution to (1) (a) Completing the square gives

$$4(x^2 + 2(1)x) + (y^2 + 2(-1)y) = 11,$$
$$4(x^2 + 2(1)x + (1)^2 - (1)^2) + (y^2 + 2(-1)y + (-1)^2 - (-1)^2) = 11,$$
$$4((x+1)^2 - 1) + ((y-1)^2 - 1) = 11,$$
$$4(x+1)^2 - 4 + (y-1)^2 - 1 = 11,$$
$$4(x+1)^2 + (y-1)^2 = 16.$$

To put this in normal form, divide both sides by 16,

$$\frac{(x+1)^2}{4} + \frac{(y-1)^2}{16} = 1,$$

$$\frac{(x-(-1))^2}{2^2} + \frac{(y-1)^2}{4^2} = 1.$$

Therefore, the ellipse is centered at $(x, y) = \boxed{(-1, 1)}$, the horizontal distance is $\boxed{2}$, and the vertical distance is $\boxed{4}$. In particular, the horizontally extreme points are $(-3, 1)$ and $(1, 1)$, whereas the vertically extreme points are $(-1, -3)$ and $(-1, 5)$.

(b) Completing the square gives

$$(x^2 + 2(1)x + 1^2 - 1^2) + 4(y^2 + 2(3)y + 3^2 - 3^2) = -36,$$

$$(x+1)^2 - 1 + 4(y+3)^2 - 36 = -36.$$

Thus the normal form is

$$\frac{(x-(-1))^2}{1^2} + \frac{(y-(-3))^2}{(1/2)^2} = 1.$$

Therefore, the ellipse is centered at $(x, y) = \boxed{(-1, -3)}$, the horizontal distance is $\boxed{1}$, and the vertical distance is $\boxed{1/2}$. In particular, the horizontally extreme points are $(-2, -3)$ and $(0, 3)$, whereas the vertically extreme points are $(-1, -7/2)$ and $(-1, -5/2)$.

(c) Completing the square gives

$$9(x^2 + 2(2)x + 2^2 - 2^2) + (y^2 + 2(-5)y + (-5)^2 - (-5)^2) + 60 = 0,$$

$$9(x+2)^2 - 36 + (y-5)^2 - 25 + 60 = 0,$$

$$9(x+2)^2 + (y-5)^2 = 1.$$

Thus the normal form is

$$\frac{(x-(-2))^2}{(1/3)^2} + \frac{(y-5)^2}{1^2} = 1.$$

Therefore, the ellipse is centered at $(x, y) = \boxed{(-2, 5)}$, the horizontal distance is $\boxed{1/3}$, and the vertical distance is $\boxed{1}$. In particular, the horizontally extreme points are $(-7/3, 5)$ and $(-5/3, 5)$, whereas the vertically extreme points are $(-2, 4)$ and $(-2, 6)$.

(d) Completing the square gives

$$9(x^2 - 2(3)x) + 4(y^2 + 2(1)y) + 49 = 0,$$

$$9(x^2 - 2(3)x + 3^2 - 3^2) + 4(y^2 + 2(1)y + 1^2 - 1^2) + 49 = 0,$$

$$9(x-3)^2 - 81 + 4(y+1)^2 - 4 + 49 = 0,$$

$$9(x-3)^2 + 4(y+1)^2 = 36.$$

To obtain the normal form, divide both sides by 36,

$$\frac{(x-3)^2}{2^2} + \frac{(y-(-1))^2}{3^2} = 1.$$

Therefore, the ellipse is centered at $(x, y) = \boxed{(3, -1)}$, the horizontal distance is $\boxed{2}$, and the vertical distance is $\boxed{3}$. In particular, the horizontally extreme points are $(1, -1)$ and $(5, -1)$, whereas the vertically extreme points are $(3, -4)$ and $(3, 2)$.

(2) In each of the following cases, find all intersection points of the given conic section and the given line.

$$(a)\ 4x^2 + 8x + y^2 - 2y = 11,\ y = 1, \quad (b)\ x^2 + y^2 = 1,\ x + y = 1, \quad (c)\ y^2 - x^2 = 1,\ y - x = 1,$$

$$(d)\ 2x + 4y^2 = 6, y + x = 3\ , \quad (e)\ x^2 + 2x + 3y^2 + 18y = 15\ 3y + x = 3.$$

Solution to (2) (a) Substituting $y = 1$ in the quadratic equation gives,

$$4x^2 + 8x + (1)^2 - 2(1) = 11,$$

$$4x^2 + 8x - 1 = 11.$$

Completing the square gives,

$$4(x^2 + 2(1)x) - 1 = 11,$$
$$4(x^2 + 2(1)x + 1^2 - 1^2) - 1 = 11,$$
$$4(x + 1)^2 - 4 - 1 = 11,$$
$$4(x + 1)^2 = 16.$$

Dividing by 4 gives,

$$(x + 1)^2 = 4.$$

Therefore, the solutions are

$$x + 1 = \pm 2, \quad x = -1 \pm 2, \quad x = -3 \text{ or } 1.$$

Thus the intersections points are $(x, y) = \boxed{(-3, 1) \text{ and } (1, 1)}$.

(b) Substituting $y = 1 - x$ into the quadratic equation gives

$$x^2 + (1 - x)^2 = 1,$$

$$x^2 + (1 - 2x + x^2) = 1,$$
$$x^2 + 1 - 2x + x^2 = 1,$$
$$2x^2 - 2x = 0,$$
$$2x(x - 1) = 0.$$

Thus, the solutions are $x = 0$ and $x = 1$. Since y equals $1 - x$, the intersection points are $(x, y) = \boxed{(0, 1)}$ and $\boxed{(1, 0)}$.

(c) Substituting $y = 1 + x$ into the quadratic equation gives

$$(1 + x)^2 - x^2 = 1,$$

$$(1 + 2x + x^2) - x^2 = 1,$$
$$1 + 2x + x^2 - x^2 = 1,$$
$$2x = 0.$$

Thus the unique solution is $x = 0$. Using that $y = 1 + x$, the unique intersection point is $(x, y) =$ $\boxed{(0, 1)}$.

(d) Substituting $x = 3 - y$ into the quadratic equation gives

$$2(3 - y) + 4y^2 = 6,$$

$$6 - 2y + 4y^2 = 6,$$
$$2y(2y - 1) = 0.$$

Thus the solutions are $y = 0$ and $y = 1/2$. Using that $x = 3 - y$, the intersection points are $(x, y) =$ $\boxed{(3, 0) \text{ and } (5/2, 1, 2)}$.

(e) Substituting $x = 3 - 3y$ into the quadratic equation gives

$$(3 - 3y)^2 + 2(3 - 3y) + 3y^2 + 18y = 15,$$

$$(9 - 18y + 9y^2) + (6 - 6y) + 3y^2 + 18y = 15,$$
$$12y^2 - 6y + 15 = 15,$$
$$6y(2y - 1) = 0.$$

Thus the solutions are $y = 0$ and $y = 1/2$. Using that $x = 3 - 3y$, the intersection points are $(x, y) =$ $\boxed{(3, 0) \text{ and } (3/2, 1/2)}$.

(3) In each of the following cases, find a pair of real numbers (x, y) with the given sum and the given product.

$$(a)\ x + y = 2,\ x \cdot y = -1, \quad (b)\ x + y = 10,\ x \cdot y = 1, \quad (c)\ x + y = 1,\ x \cdot y = -1.$$

Solution to (3) (a) From the first equation, y equals $2 - x$. Substituting $y = 2 - x$ into the second equation gives

$$x(2 - x) = -1,$$
$$-x^2 + 2x = -1,$$
$$x^2 - 2x + 1 = 0,$$
$$(x - 1)^2 = 0.$$

Thus the unique solution is $x = -1$. Since y equals $2 - x$, the unique solution is $(x, y) = \boxed{(1, 1)}$.

(b) From the first equation, y equals $10 - x$. Substituting $y = 10 - x$ into the second equation gives

$$x(10 - x) = 1,$$
$$-x^2 + 10x = 1,$$
$$x^2 - 10x + 1 = 0,$$
$$(x^2 - 2(5)x) + 1 = 0,$$
$$(x^2 - 2(5)x + 5^2 - 5^2) + 1 = 0,$$
$$(x - 5)^2 - 25 + 1 = 0,$$
$$(x - 5)^2 = 24.$$

Thus the solutions are

$$x = 5 \pm 2\sqrt{6}.$$

Using that $y = 10 - x$, the solutions are $(x, y) = \boxed{(5 - 2\sqrt{6}, 5 + 2\sqrt{6}) \text{ and } (5 + 2\sqrt{6}, 5 - 2\sqrt{6})}$.

(c) From the first equation, y equals $1 - x$. Substituting $y = 1 - x$ into the second equation gives

$$x(1 - x) = -1,$$
$$-x^2 + x = -1,$$
$$x^2 - x + 1 = 0,$$
$$(x^2 + 2(-1/2)x) + 1 = 0,$$
$$(x^2 + 2(-1/2)x + (-1/2)^2 - (-1/2)^2) + 1 = 0,$$
$$(x - 1/2)^2 - 1/4 + 1 = 0,$$
$$(x - 1/2)^2 = -3/4.$$

Of course $\boxed{\text{there are no real solutions}}$ of this equation.

(4) Find all points on the line $y + x = 2$ whose distance from the origin equals $3/2$.

Solution to (4) For a point (x, y) in the plane, the distance to the origin is $\sqrt{x^2 + y^2}$. Thus, the second equation is

$$\sqrt{x^2 + y^2} = 3/2.$$

Since both sides are nonnegative real numbers, this equation is equivalent to

$$x^2 + y^2 = (3/2)^2 = 9/4.$$

Since y equals $2 - x$, this gives

$$x^2 + (2 - x)^2 = 9/4,$$

$$x^2 + 4 - 4x + x^2 = 9/4,$$
$$2x^2 - 4x + 4 = 9/4,$$
$$2(x^2 + 2(-1)x) + 4 = 9/4,$$
$$2(x^2 + 2(-1)x + (-1)^2 - (-1)^2) + 4 = 9/4,$$
$$2(x-1)^2 - 2 + 4 = 9/4,$$
$$2(x-1)^2 = 1/4,$$
$$(x-1)^2 = 2/16.$$

The solutions are

$$x = 1 \pm \sqrt{2}/4.$$

Since y equals $2 - x$, the solutions are $(x, y) = \boxed{(1 - \sqrt{2}/4, 1 + \sqrt{2}/4) \text{ and } (1 + \sqrt{2}/4, 1 - \sqrt{2}/4)}$.

(5) For each of the following functions, state the maximal domain and range of the function. For the given interval, state whether the restriction of the function to that interval is invertible. Whenever the restriction function is invertible, find a formula for the inverse function, and state the domain and range of the inverse function.

$$(a)\ f(x) = 2x^{1/2} + 1,\ [0,4],\quad (b)\ g(x) = 7x^3 - 7,\ (-2,+2),$$

$$(c)\ h(x) = 3x^{2/3} - 6x^{1/3},\ [0,1],\quad (d)\ k(x) = 3x^{2/3} - 6x^{1/3},\ [0,8].$$

Solution to (5) (a) Since $x^{1/2}$ is only defined for nonnegative real numbers, the maximal domain is $\boxed{[0,\infty)}$. Since $x^{1/2} \geq 0$, also $2x^{1/2} + 1 \geq 0$. Thus the maximal range is $\boxed{[1,\infty)}$. On the specified domain, $[0,4]$, the range of the restricted function is $[1,5]$. To find an inverse function, we solve for x the equation

$$y = 2x^{1/2} + 1,$$
$$y - 1 = 2x^{1/2},$$
$$(y-1)/2 = x^{1/2}.$$

This is solvable on the specifed interval $[0,4]$, in fact on any subset of $[0,\infty)$, thus the restriction function is $\boxed{\text{invertible}}$, and the inverse function is

$$\boxed{f^{-1}(y) = (y-1)^2/4}.$$

Since the range of the restricted function is $[1,5]$, the domain of the inverse function is $\boxed{[1,5]}$. Since the domain of the restricted function is $[0,4]$, the range of the inverse function is $\boxed{[0,4]}$.

(b) The maximal domain of $g(x)$ is $\boxed{(-\infty,\infty)}$. The maximal range is also $\boxed{(-\infty,\infty)}$. On the specified interval, the restricted function has range $(-63,49)$. To find an inverse function, solve for x the equation

$$y = 7x^3 - 7,$$
$$y + 7 = 7x^3,$$
$$(y+7)/7 = x^3.$$

This is solvable for x in $(-2,2)$, in fact for x in any subset of $(-\infty,+\infty)$. Thus the restricted function is $\boxed{\text{invertible}}$, and the inverse function is

$$\boxed{g^{-1}(y) = (y+7)^{1/3}/7^{1/3}}.$$

Since the range of the restriction function is $(-63,49)$, the domain of the inverse function is $\boxed{(-63,49)}$. Since the domain of the restriction function is $(-2,2)$, the range of the inverse function is $\boxed{(-2,2)}$.

(c) The maximal domain of $h(x)$ is all of $\boxed{(\infty,\infty)}$. Since

$$h(x) = 3((x^{1/3})^2 - 2x^{1/3} + 1) - 3 = 3(x^{1/3}-1)^2 - 3,$$

the maximal range is $\boxed{(-3,\infty)}$. To try to find an inverse function, solve for x the equation

$$y = 3(x^{1/3})^2 - 6x^{1/3} = 3(x^{1/3}-1)^2 - 3,$$
$$y + 3 = 3(x^{1/3}-1)^2,$$
$$(x^{1/3}-1)^2 = (3y+9)/9.$$

This is only solvable on an interval where either $x^{1/3} \geq 1$ or $x^{1/3} \leq 1$, i.e., on a subinterval of $[1,\infty)$ or a subinterval of $(-\infty,1]$. Thus, the restricted function is $\boxed{\text{invertible}}$, and the inverse function is given by

$$\boxed{h^{-1}(y) = (-\sqrt{3y+9}/3 + 1)^3}.$$

On $[0,1]$, the range of $h(x)$ is $[-3,0]$. Thus the domain of the inverse function is $\boxed{[-3,0]}$. Since the domain of the restricted function is $[0,1]$, the range of the inverse function is $\boxed{[0,1]}$.

(d) Since the interval $[0,8]$ is neither a subinterval of $(-\infty,1]$ nor a subinterval of $[1,\infty)$, the restricted function $k(x)$ is $\boxed{\text{not invertible}}$. Explicitly, $k(0)$ equals $k(8)$, so that the restricted function is not one-to-one.

(6) In each of the following cases, find all zeroes of the given polynomial function, determine the sign of the function between the zeroes, and find the behavior at $\pm\infty$. Also, find the y-intercept, and say whether the function is even, odd or neither. Give a rough sketch of the graph illustrating all of these features.

(a) $f(x) = x(x^2-1)$, (b) $g(x) = x^3-9x$, (c) $h(x) = x^4-1$, (d) $k(x) = x^4-5x^2+4$, (e) $l(x) = x^4-5x^3+4x^2$.

Solution to (6) (a) Since $x^2 - 1$ factors as $(x-1)(x+1)$, the function is

$$f(x) = (x+1)x(x-1).$$

Note first that this is an odd function. Also the zeroes are precisely $x =$ −1, 0, and 1. Testing a point on each side, $f(x)$ is negative on $(-\infty, -1)$, $f(x)$ is positive on $(-1, 0)$, $f(x)$ is negative on $(0, 1)$, and $f(x)$ is positive on $(1, \infty)$. As x approaches $-\infty$, $f(x)$ approaches $-\infty$. As x approaches $+\infty$, $f(x)$ approaches $+\infty$. As an odd function, $f(0)$ equals 0.

(b) Factoring gives $g(x) = x(x^2-9)$. Since (x^2-9) factors as $(x-3)(x+3)$, the function is

$$g(x) = (x+3)x(x-3).$$

Note first that this is an odd function. Also the zeroes are precisely $x =$ −3, 0, and 3. Testing a point on each side, $f(x)$ is negative on $(-\infty, -3)$, $f(x)$ is positive on $(-3, 0)$, $f(x)$ is negative on $(0, 3)$, and $f(x)$ is positive on $(3, \infty)$. As x approaches $-\infty$, $f(x)$ approaches $-\infty$. As x approaches ∞, $f(x)$ approaches ∞. As an odd function, $f(0)$ equals 0.

(c) As a difference of squares, $x^4 - 1$ factors as $(x^2-1)(x^2+1)$. Of course $x^2 + 1$ is always ≥ 1, hence it cannot factor as a product of linear polynomials (or else it would have zeroes). Finally, $x^2 - 1$ factors as $(x-1)(x+1)$. Thus the function is

$$h(x) = (x+1)(x^2+1)(x-1).$$

Note first that this is an even function. Also the zeroes are precisely $x =$ −1 and 1. Testing a point on each side gives that $f(x)$ is positive on $(-\infty, -1)$, $f(x)$ is negative on $(-1, 1)$, and $f(x)$ is positive on $(1, \infty)$. As x approaches $-\infty$, $f(x)$ approaches ∞. As x approaches ∞, $f(x)$ approaches ∞. Plugging in, $f(0)$ equals -1.

(d) Since $t^2 - 5t + 4$ factors as $(t-4)(t-1)$, also $x^4 - 5x^2 + 4$ factors as $(x^2-4)(x^2-1)$. Since $x^2 - 1$ factors as $(x-1)(x+1)$ and since $x^2 - 4$ factors as $(x-2)(x+2)$, the function is

$$k(x) = (x+2)(x+1)(x-1)(x-2).$$

Note first that this is an even function. Also the zeroes are precisely $x =$ −2, −1, 1, and 2. Testing a point on each side, $f(x)$ is positive on $(-\infty, -2)$, $f(x)$ is negative on $(-2, -1)$, $f(x)$ is positive on $(-1, 1)$, $f(x)$ is negative on $(1, 2)$, and $f(x)$ is positive on $(2, \infty)$. As x approaches $-\infty$, $f(x)$ approaches ∞. As x approaches ∞, $f(x)$ approaches ∞. Plugging in, $f(0)$ equals 4.

(e) Factoring gives $l(x) = x^2(x^2 - 5x + 4)$. Since $x^2 - 5x + 4$ factors as $(x-4)(x-1)$, the function is

$$l(x) = x^2(x-1)(x-4).$$

This function is $\boxed{\text{neither even nor odd}}$. The zeroes are precisely $x = \boxed{0, 1, \text{ and } 4}$. Testing a point on each side, $f(x)$ is positive on $(-\infty, 0)$, $f(x)$ is positive on $(0,1)$, $f(x)$ is negative on $(1,4)$, and $f(x)$ is positive on $(4,\infty)$. As x approaches $-\infty$, $f(x)$ approaches $\boxed{\infty}$. As x approaches ∞, $f(x)$ approaches $\boxed{\infty}$. Plugging in, $f(0)$ equals $\boxed{4}$.

(7) Perform each of the following computations.

$$(a) f(x) = x^3 + 2x,\ (f(x+1) - f(1))/x =?,\ \ (b) f(x) = x^2 + 1,\ g(x) = x^2 - 1,\ f(g(x)) =?,$$

$$(c) f(x) = x^2 + 1,\ g(x) = x^2 - 1,\ g(f(x)) =?,\ \ (d) f(x) = 2x + 1,\ (f(x))^3 =?.$$

Solution to (7) (a) First, $f(x+1)$ equals

$$(x+1)^3 + 2(x+1) = (x^3 + 3x^2 + 3x + 1) + (2x + 2) = x^3 + 3x^2 + 5x + 3.$$

Also, $f(1)$ equals $1^3 + 2(1) = 1 + 2 = 3$. Thus,

$$f(x+1) - f(1) = (x^3 + 3x^2 + 5x + 3) - 3 = x^3 + 3x^2 + 5x.$$

Thus, factoring out x gives,

$$(f(x+1) - f(1))/x = (x^3 + 3x^2 + 5x)/x = \boxed{x^2 + 3x + 5}.$$

(b) First, $(g(x))^2$ equals

$$(x^2 - 1)^2 = x^4 - 2x^2 + 1.$$

Thus,

$$f(g(x)) = (g(x))^2 + 1 = (x^4 - 2x^2 + 1) + 1 = \boxed{x^4 - 2x^2 + 2}.$$

(c) First, $(f(x))^2$ equals

$$(x^2 + 1)^2 = x^4 + 2x^2 + 1.$$

Thus,

$$g(f(x)) = (x^2 + 1)^2 - 1 = (x^4 + 2x^2 + 1) - 1 = \boxed{x^4 + 2x^2}.$$

(d) By direct computation,

$$(2x + 1)^3 = (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3 = \boxed{8x^3 + 12x^2 + 6x + 1}.$$

(8) In each of the following cases, state the maximal domain and range of the given function. For the restriction of the function to the specified interval, state whether or not the restricted function is invertible. If the restricted function is invertible, find the inverse function. State the domain and range of the inverse function.

$$(a)\ f(x) = 8(2^x) - 1,\ [-3, 3],\ \ (b)\ f(x) = (8(2^x) - 1)^{-1},\ (-3, 3),$$

$$(c)\ f(x) = 3^{2x} - 2(3^x) + 1,\ [0,2],\quad (d)\ f(x) = 3^{2x} - 2(3^x) + 1,\ [-1,2].$$

$$(e)\ f(x) = \log_5(x-1),\ [2,26],\quad (f)\ f(x) = \log_5(x^2 - 2x + 1),\ [2,26],$$
$$(g)\ f(x) = 1/\log_5(x-1),\ (2,26),\quad (h)\ f(x) = \log_2(x+1) - \log_2(x-1),\ (1,3).$$

Solution to (8) (a) The maximal domain is $\boxed{(-\infty,\infty)}$. The maximal range is $\boxed{(-1,\infty)}$. To find an inverse function, solve for x the equation

$$y = 8(2^x) - 1,$$
$$y + 1 = 8(2^x),$$
$$(y+1)/8 = 2^x.$$

On the interval $[-3,3]$, in fact on any subinterval of $(-\infty,\infty)$, this function is $\boxed{\text{invertible}}$. The inverse of the restricted function is

$$f^{-1}(y) = \boxed{\log_2((y+1)/8)}.$$

Since the range of the restricted function is $[0,63]$, the domain of the inverse function is $\boxed{[0,63]}$. Since the domain of the restricted function is $[-3,3]$, the range of the inverse function is $\boxed{[-3,3]}$.

(b) The denominator equals 0 when x equals -3. Thus the maximal domain is $\boxed{(-\infty,-3)\cup(-3,\infty)}$. On the interval $(-\infty,-3)$, the range is $\boxed{(-\infty,-1)}$. On the interval $(-3,\infty)$, the range is $\boxed{(0,\infty)}$. Altogether, the maximal range is $\boxed{(-\infty,-1)\cup(0,\infty)}$. To find an inverse function, solve for x the equation

$$y = 1/(8(2^x) - 1),$$
$$1/y = 8(2^x) - 1,$$
$$(1 + (1/y))/8 = 2^x.$$

On the interval $(-3,3)$, in fact on any subinterval of $(-\infty,-3)\cup(-3,\infty)$, this function is $\boxed{\text{invertible}}$. The inverse of the restricted function is

$$f^{-1}(y) = \boxed{\log_2((1 + (1/y))/8)}.$$

Since the range of the restricted function is $(1/63,\infty)$, the domain of the inverse function is $\boxed{(1/63,\infty)}$. Since the domain of the restricted function is $(-3,3)$, the range of the inverse function is $\boxed{(-3,3)}$.

(c) The maximal domain is $\boxed{(-\infty,\infty)}$. Since

$$f(x) = (3^x - 1)^2,$$

the range on the interval $(-\infty, 0]$ equals $\boxed{[0,1)}$, and the range on the interval $[0,\infty)$ equals $\boxed{[0,\infty)}$, so that the maximal range is $\boxed{[0,\infty)}$. On the interval $[0,2]$, in fact on any subinterval of $(-\infty,0]$ and on any subinterval of $[0,\infty)$, the function is $\boxed{\text{invertible}}$. The inverse of the restricted function is

$$\boxed{f^{-1}(y) = \log_3(y^{1/2}+1)}.$$

Since the range of the restricted function is $[0,64]$, the domain of the inverse function is $\boxed{[0,64]}$. Since the domain of the restricted function is $[0,2]$, the range of the inverse function is $\boxed{[0,2]}$.

(d) Since $[-1,2]$ is not a subinterval of $(-\infty,0]$ nor a subinterval of $[0,\infty)$, the restricted function is $\boxed{\text{not invertible}}$.

(e) The maximal domain is $\boxed{(1,\infty)}$. The maximal range is $\boxed{(-\infty,\infty)}$. To find an inverse function, solve for x the equation

$$y = \log_5(x-1),$$
$$5^y = x-1,$$
$$5^y + 1 = x.$$

On the interval $[2,26]$, in fact on any subinterval of $(1,\infty)$, this function is $\boxed{\text{invertible}}$. The inverse of the restricted function is

$$f^{-1}(y) = \boxed{5^y+1}.$$

Since the range of the restricted function is $[0,2]$, the domain of the inverse function is $\boxed{[0,2]}$. Since the domain of the restricted function is $[2,26]$, the range of the inverse function is $\boxed{[2,26]}$.

(f) The maximal domain is $\boxed{(1,\infty)}$. The maximal range is $\boxed{(-\infty,\infty)}$. To find an inverse function, solve for x the equation

$$y = \log_5((x-1)^2) = 2\log_5(x-1),$$
$$y/2 = \log_5(x-1),$$
$$5^{y/2} = x-1.$$

On the interval $[2,26]$, in fact on any subinterval of $(1,\infty)$, this function is $\boxed{\text{invertible}}$. The inverse of the restricted function is

$$f^{-1}(y) = \boxed{5^{y/2}+1}.$$

Since the range of the restricted function is $[0,4]$, the domain of the inverse function is $\boxed{[0,4]}$. Since the domain of the restricted function is $[2,26]$, the range of the inverse function is $\boxed{[2,26]}$.

(g) The denominator equals 0 if x equals 2. Thus the maximal domain is $\boxed{(1,2)\cup(2,\infty)}$. The range on $(1,2)$ is $\boxed{(-\infty,0)}$, and the range on $(2,\infty)$ is $\boxed{(0,\infty)}$, so that the maximal range is

$(-\infty, 0) \cup (0, \infty)$. To find an inverse function, solve for x the equation

$$y = 1/\log_5(x-1),$$

$$1/y = \log_5(x-1),$$

$$x = 5^{1/y} + 1.$$

On the interval $(2, 26)$, in fact on any subset of $(1, 2) \cup (2, \infty)$, this function is invertible. The inverse of the restricted function is

$$f^{-1}(y) = 5^{1/y} + 1.$$

Since the range of the restricted function is $(1/2, \infty)$, the domain of the inverse function is $(1/2, \infty)$. Since the domain of the restricted function is $(2, 26)$, the range of the inverse function is $(2, 26)$.

(h) The maximal domain is $(1, \infty)$. The maximal range is $(0, \infty)$. To find an inverse function, solve for x the equation

$$y = \log_2(x+1) - \log_2(x-1) = \log_2((x+1)/(x-1)),$$

$$2^y = \frac{x+1}{x-1},$$

$$2^y(x-1) = (x+1),$$

$$2^y x - 2^y = x + 1,$$

$$(2^y - 1)x = 2^y + 1.$$

On the interval $(1, 3)$, in fact on any subinterval of $(1, \infty)$, this function is invertible. The inverse of the restricted function is

$$f^{-1}(y) = (2^y + 1)/(2^y - 1).$$

Since the range of the restricted function is $(1, \infty)$, the domain of the inverse function is $(1, \infty)$. Since the domain of the restricted function is $(1, 3)$, the range of the inverse function is $(1, 3)$.

(9) A sample of bacteria grows from one milligram to eight milligrams in nine hours. Assuming the bacteria follows an exponential growth model, determine the doubling time. Also determine the total amount of time necessary for the sample to grow from one milligram to sixty-four milligrams.

Solution to (9) The exponential growth model is

$$a(t) = a_0 2^{t/d},$$

where a_0 is the initial mass and where d is the doubling time. Since a_0 equals 1 mg, and since $a(9 \text{ hrs })$ equals 8 mg, we have the equation,

$$8 \text{ mg } = (1 \text{ mg })2^{9 \text{ hrs } /d},$$

$$8 = 2^{9 \text{ hrs } /d}.$$

Since 8 equals 2^3, the unique solution is

$$9 \text{ hrs } /d = 3,$$

$$3d = 9 \text{ hrs },$$

$$d = 9 \text{ hrs } /3.$$

Thus the doubling time is 3 hours. Since 64 equals 2^6, the solution of

$$64 \text{ mg } = (1 \text{ mg })2^{t/3 \text{ hrs}}$$

is

$$t/(3 \text{ hrs }) = 6,$$

$$t = 18 \text{ hours}.$$

(10) The half-life of a certain radioactive isotope is 5000 years. Assuming the radioactive decay follows an exponential decay model, how long is necessary for a quantity of 24 kilograms of the isotope to decay to 6 kilograms? How long is necessary for the quantity to decay to 3/4 of a kilogram?

Solution to (10) The exponential decay model is

$$a(t) = a_0(1/2)^{t/h},$$

where a_0 is the initial mass, and where h is the half-life. Thus, the equation to solve is

$$6 \text{ kg } = (24 \text{ kg })(1/2)^{t/5000 \text{ yrs}},$$

$$(1/2)^2 = (1/2)^{t/5000 \text{ yrs}}.$$

The unique solution is

$$t/5000 \text{ yrs } = 2,$$

$$t = 10000 \text{ yrs}.$$

Similarly, for decay to 3/4 kg , the equation to solve is

$$(3/4) \text{ kg } = (24 \text{ kg })(1/2)^{t/5000 \text{ yrs}},$$

$$(1/2)^5 = (1/2)^{t/5000 \text{ yrs}}.$$

The unique solution is

$$t/5000 \text{ yrs} = 5,$$

$$t = 25000 \text{ yrs}.$$

(11) Recall that measurements on the Richter scale are a common logarithm of the amplitude S of vibration divided by a calibration amplitude S_0. If a seismic event ranks 6 on the Richter scale and an aftershock ranks 2 on the Richter scale, how much stronger is the amplitude of the original event than the amplitude of the aftershock?

Solution to (11) The Richter scale model is

$$R = \log_{10}(S/S_0),$$

where S is the amplitude of vibration, where S_0 is a calibration amplitude, and where R is the Richter scale value. For the initial seismic event,

$$\log_{10}(S_i/S_0) = 6.$$

For the aftershock,

$$\log_{10}(S_a/S_0) = 2.$$

Thus,

$$4 = 6 - 2 = \log_{10}(S_i/S_0) - \log_{10}(S_a/S_0) = \log_{10}((S_i/S_0)/(S_a/S_0)) = \log_{10}(S_i/S_a).$$

Therefore,

$$10^4 = S_i/S_a,$$

i.e., the amplitude of the original event is 10000 time stronger than the amplitude of the aftershock.