MAT 123 Introduction to Calculus — **Jason Starr**
Final Exam — **Spring 2015**
Wednesday 5/13 8:00am

MAT 123 Practice for Final Exam with Solutions

Remark. The final exam will be **cumulative**. Please consult the review sheets for Midterms 1 and 2 in addition to this review sheet. If you are comfortable with the problems on all of the review sheets, you should be well prepared for the mastery problems on the final exam.

Exam Policies. You must show up on time for all exams. Please bring your student ID card: ID cards may be checked, and students may be asked to sign a picture sheet when turning in exams. Other policies for exams will be announced / repeated at the beginning of the exam.

If you have a university-approved reason for taking an exam at a time different than the scheduled exam (because of a religious observance, a student-athlete event, etc.), please contact your instructor as soon as possible. Similarly, if you have a documented medical emergency which prevents you from showing up for an exam, again contact your instructor as soon as possible.

All exams are closed notes and closed book. Once the exam has begun, having notes or books on the desk or in view will be considered cheating and will be referred to the Academic Judiciary.

It is not permitted to use cell phones, calculators, laptops, MP3 players, Blackberries or other such electronic devices at any time during exams. If you use a hearing aid or other such device, you should make your instructor aware of this before the exam begins. You must turn off your cell phone, etc., prior to the beginning of the exam. If you need to leave the exam room for any reason before the end of the exam, it is still not permitted to use such devices. Once the exam has begun, use of such devices or having such devices in view will be considered cheating and will be referred to the Academic Judiciary. Similarly, once the exam has begun any communication with a person other than the instructor or proctor will be considered cheating and will be referred to the Academic Judiciary.

Mastery Review Topics.
In addition to the Core Competency Exams, the final exam will include mastery questions. Students who have already passed the Core Competency Exams need not repeat the Core Competency Exam: those students may devote all of their time and energy to the mastery questions. For the final exam, these are the mastery topics that are not included on the earlier review sheets.

(1) Know how to convert between radian angle measure and degree angle measure.

(2) For a shifted and scaled graph of cosine, find the amplitude, the period, the phase shift and the average value of the corresponding function. Conversely, if given the average value, the period, the amplitude and the phase shift, be able to write the formula for the shifted cosine function, and be able to graph several periods of this function.

(3) Know the double-angle formulas. Use the double-angle formulas to find the values of cosine and sine for the double of an angle whose cosine and sine are given. Conversely, if given the value of cosine of the double of an angle, find a quadratic equation satisfied by the cosine of the angle.

(4) Know the angle addition formulas. Use the angle addition formulas to find the values of cosine and sine for the sum of two angles whose cosine and sine values are given.

(5) Confirm stated identities among trigonometric functions, including cotangent, secant and cosecant, using the Pythagorean identity, the double-angle formulas, the angle addition formulas, etc.

(6) Know the definitions of the inverse trigonometric function, including the domain and range. Recognize the graphs of the inverse trigonometric functions. Know special values of the inverse trigonometric functions.

(7) Know the basic identities satisfied by inverse trigonometric functions. If given the value of an inverse trigonometric function, be able to find all angle measures on which the trigonometric function takes the same value.

(8) For a right triangle, if given the lengths of two sides, be able to find the length of the remaining side and the (interior) angles. Similarly, if given the length of one side and given one angle, be able to find the lengths of the remaining sides.

(9) For polynomial functions $p(x)$ and $q(x)$, find the domain of the rational function $p(x)/q(x)$. Be able to add, subtract, multiply and divide such functions, with special attention to the new function.

(10) Perform polynomial division with remainder. Simplify an improper fraction of polynomials to the sum of a polynomial and a proper fraction of polynomials (the degree of the numerator is less than the degree of the denominator).

(11) Identify certain key features of the graph of a rational function: the domain, the horizontal asymptote (if it exists), and all vertical asymptotes (if any exist).

Practice Problems.

(1) In each of the following cases, convert the given degree measure of an angle to the corresponding radian measure.

$(a)\ 30°$, $(b)\ 75°$, $(c)\ -120°$, $(d)\ 200°$, $(e)\ (200/\pi)°$, $(f)\ 285°$, $(g)\ -720°$, $(h)\ 135°$.

(2) In each of the following cases, convert the given radian measure of an angle to the corresponding degree measure.

$$(a)\ \pi/6,\quad (b)\ 5\pi/12,\quad (c)\ -2\pi/3,\quad (d)\ 10\pi/9,\quad (e)\ 10/9,\quad (f)\ 19\pi/12,\quad (g)\ -4\pi,\quad (h)\ 3\pi/4.$$

Solutions to (1) and (2) The solution to each part of (1) is the corresponding part of (2), and vice versa.

(3) In each of the following cases, identify the average value of the function (i.e., the midpoint between the maximum and minimum values), the period, the amplitude and the phase shift.

$$(a)\ 2\cos(3x),\quad (b)\ (1/2)+(1/2)\cos(2x-\pi/2),\quad (c)\ 5+10\cos((x/2)-\pi),$$

$$(d)\ (-1/\sqrt{2})+\cos(x-(\pi/4)),\quad (e)\ \sin(x),\quad (f)\ (1/\sqrt{2})\sin(x)+(1/\sqrt{2})\cos(x).$$

Solution to (3) (a) The maximum occurs at $x = 0, \pm 2\pi/3, \pm 4\pi/3$, etc., and $y_{\max} = 2$. The minimum occurs at $x = \pi/3, \pi/3 \pm 2\pi/3, \pi/3 \pm 4\pi/3$, etc., and $y_{\min} = -2$. Thus the average value is $y_0 = (y_{\max} + y_{\min})/2 = 0$. Moreover, the amplitude is $A = (y_{\max} - y_{\min})/2 = 2$. The period is $p = 2\pi/3$. Finally, the phase shift is $\phi = 0$.

(b) The maximum occurs at $x = \pi/4, \pi/4 \pm \pi, \pi/4 \pm 2\pi$, etc., and $y_{\max} = 1$. The minimum occurs at $x = 3\pi/4, 3\pi/4 \pm \pi, 3\pi/4 \pm 2\pi$, etc., and $y_{\min} = 0$. Thus the average value is $y_0 = (y_{\max} + y_{\min})/2 = 1/2$. Moreover, the amplitude is $A = (y_{\max} - y_{\min})/2 = 1/2$. The period is $p = \pi$. Finally, the phase shift is $\phi = \pi/4$.

(c) The maximum occurs at $x = 2\pi, 2\pi \pm 4\pi, 2\pi \pm 8\pi$ etc., and $y_{\max} = 15$. The minimum occurs at $x = 0, \pm 4\pi, \pm 8\pi$, etc., and $y_{\min} = -5$. Thus the average value is $y_0 = (y_{\max} + y_{\min})/2 = 5$. Moreover, the amplitude is $A = (y_{\max} - y_{\min})/2 = 10$. The period is $p = 4\pi$. Finally, the phase shift is $\phi = 2\pi$.

(d) The maximum occurs at $x = \pi/4, \pi/4 \pm 2\pi, \pi/4 \pm 4\pi$ etc., and $y_{\max} = (2 - \sqrt{2})/2$. The minimum occurs at $x = \pi/3, \pi/3 \pm 2\pi, \pi/3 \pm 4\pi$, etc., and $y_{\min} = -(2 + \sqrt{2})/2$. Thus the average value is $y_0 = (y_{\max} + y_{\min})/2 = -\sqrt{2}/2$. Moreover, the amplitude is $A = (y_{\max} - y_{\min})/2 = 1$. The period is $p = 2\pi$. Finally, the phase shift is $\phi = \pi/4$.

(e) The maximum occurs at $x = \pi/2, \pi/2 \pm 2\pi, \pi/2 \pm 4\pi$, etc., and $y_{\max} = 1$. The minimum occurs at $x = 3\pi/2, 3\pi/2 \pm 2\pi, 3\pi/2 \pm 4\pi$, etc., and $y_{\min} = -1$. Thus the average value is $y_0 = (y_{\max} + y_{\min})/2 = 0$. Moreover, the amplitude is $A = (y_{\max} - y_{\min})/2 = 1$. The period is $p = 2\pi$. Finally, the phase shift is $\phi = \pi/2$.

(f) By the angle addition formulas,

$$(1/\sqrt{2})\sin(x) + (1/\sqrt{2})\cos(x) = \cos(-\pi/4)\cos(x) - \sin(-\pi/4)\sin(x) = \cos(x - \pi/4).$$

The maximum occurs at $x = \pi/4, \pi/4 \pm 2\pi, \pi/4 \pm 4\pi$, etc., and $y_{\max} = 1$. The minimum occurs at $x = 5\pi/4, 5\pi/4 \pm 2\pi, 5\pi/4 \pm 4\pi$, etc., and $y_{\min} = -1$. Thus the average value is $y_0 = (y_{\max} + y_{\min})/2 =$

0. Moreover, the amplitude is $A = (y_{\max} - y_{\min})/2 = 1$. The period is $p = 2\pi$. Finally, the phase shift is $\pi/4$.

(4) In each of the following cases from **Problem (3)**, sketch several periods of the graph. Please label the x-coordinate and the y-coordinate of each maximum value and of each minimum value on your graph. Next, shuffle your graphs and derived the formula for each graph only from the information of the x-coordinate and y-coordinate of two consecutive maximum values and a minimum value.

(5) In each case, the cosine and sine of an angle θ is given. Compute $\cos(2\theta)$ and $\sin(2\theta)$.

$$(a)\ \cos(\theta) = 3/5, \sin(\theta) = 4/5, \cos(2\theta) = ?, \sin(2\theta) = ?,$$

$$(b)\ \cos(\theta) = 4/5, \sin(\theta) = -3/5, \cos(2\theta) = ?, \sin(2\theta) = ?,$$

$$(c)\ \cos(\theta) = -5/13, \sin(\theta) = 12/13, \cos(2\theta) = ?, \sin(2\theta) = ?,$$

$$(d)\ \cos(\theta) = \sqrt{2+\sqrt{2}}/2, \sin(\theta) = \sqrt{2-\sqrt{2}}/2, \cos(2\theta) = ?, \sin(2\theta) = ?.$$

Solution to (5) (a) By the double-angle formula,

$$\cos(2\theta) = (\cos(\theta))^2 - (\sin(\theta))^2 = 9/25 - 16/25 = \boxed{-7/25},$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2(4/5)(3/5) = \boxed{24/25}.$$

As a double-check, notice that

$$(\cos(2\theta))^2 + (\sin(2\theta))^2 = 49/625 + 576/625 = 1.$$

(b) By the double-angle formula,

$$\cos(2\theta) = (\cos(\theta))^2 - (\sin(\theta))^2 = 16/25 - 9/25 = \boxed{7/25},$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2(-3/5)(-4/5) = \boxed{-24/25}.$$

As a double-check, notice that

$$(\cos(2\theta))^2 + (\sin(2\theta))^2 = 576/625 + 49/625 = 1.$$

(c) By the double-angle formula,

$$\cos(2\theta) = (\cos(\theta))^2 - (\sin(\theta))^2 = 25/169 - 144/169 = \boxed{-119/169},$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2(-5/13)(12/13) = \boxed{-120/169}.$$

As a double-check, notice that

$$(\cos(2\theta))^2 + (\sin(2\theta))^2 = 14161/28561 + 14400/28561 = 1.$$

(d) By the double-angle formula,

$$\cos(2\theta) = (\cos(\theta))^2 - (\sin(\theta))^2 = (2+\sqrt{2})/4 - (2-\sqrt{2})/4 = \boxed{\sqrt{2}/2},$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\sqrt{(2+\sqrt{2})(2-\sqrt{2})}/4 = \boxed{\sqrt{2}/2}.$$

As a double-check, notice that

$$(\cos(2\theta))^2 + (\sin(2\theta))^2 = 2/4 + 2/4 = 1.$$

(6) In each case, the cosine of 2θ is given and an interval containing θ is given. Find a quadratic equation satisfied by $\cos(\theta)$ and $\sin(\theta)$, and then solve each equation.

$$(a)\ \cos(2\theta) = \sqrt{2}/2, 0 \le \theta < \pi/2, \quad (b)\ \cos(2\theta) = 3/4, -\pi/2 \le \theta < 0,$$

$$(c)\ \cos(2\theta) = 1/2, 0 < \theta \le \pi/2, \quad (d)\ \cos(2\theta) = \sqrt{2+\sqrt{2}}/2, 0 \le \theta < \pi/2,$$
$$(e)\ \cos(2\theta) = 1, \pi/2 < \theta \le \pi.$$

Solution to (6) (a) By the double-angle formula,

$$\frac{\sqrt{2}}{2} = \cos(2\theta) = 2(\cos(\theta))^2 - 1 = 1 - 2(\sin(\theta))^2.$$

Thus, we have the quadratic equations,

$$(\cos(\theta))^2 = \frac{2+\sqrt{2}}{4}, \quad (\sin(\theta))^2 = \frac{2-\sqrt{2}}{4}.$$

Since $0 \le \theta < \pi/2$, both $\sin(\theta)$ and $\cos(\theta)$ are positive. Thus the solutions are

$$\cos(\theta) = \boxed{\sqrt{2+\sqrt{2}}/2}, \quad \sin(\theta) = \boxed{\sqrt{2-\sqrt{2}}/2}.$$

(b) By the double-angle formula,

$$\frac{3}{4} = \cos(2\theta) = 2(\cos(\theta))^2 - 1 = 1 - 2(\sin(\theta))^2.$$

Thus, we have the quadratic equations,

$$(\cos(\theta))^2 = \frac{14}{16}, \quad (\sin(\theta))^2 = \frac{2}{16}.$$

Since $-\pi/2 \leq \theta < 0$, $\cos(\theta)$ is positive and $\sin(\theta)$ is negative. Thus the solutions are

$$\cos(\theta) = \boxed{\sqrt{14}/4}, \quad \sin(\theta) = \boxed{-\sqrt{2}/4}.$$

(c) By the double-angle formula,

$$\frac{1}{2} = \cos(2\theta) = 2(\cos(\theta))^2 - 1 = 1 - 2(\sin(\theta))^2.$$

Thus, we have the quadratic equations,

$$(\cos(\theta))^2 = \frac{3}{4}, \quad (\sin(\theta))^2 = \frac{1}{4}.$$

Since $0 < \theta \leq \pi/2$, both $\sin(\theta)$ and $\cos(\theta)$ are positive. Thus the solutions are

$$\cos(\theta) = \boxed{\sqrt{3}/2}, \quad \sin(\theta) = \boxed{1/2}.$$

(d) By the double-angle formula,

$$\frac{\sqrt{2+\sqrt{2}}}{2} = \cos(2\theta) = 2(\cos(\theta))^2 - 1 = 1 - 2(\sin(\theta))^2.$$

Thus, we have the quadratic equations,

$$(\cos(\theta))^2 = \frac{2+\sqrt{2+\sqrt{2}}}{4}, \quad (\sin(\theta))^2 = \frac{2-\sqrt{2+\sqrt{2}}}{4}.$$

Since $0 < \theta \leq \pi$, both $\sin(\theta)$ and $\cos(\theta)$ are positive. Thus the solutions are

$$\cos(\theta) = \boxed{\sqrt{2+\sqrt{2+\sqrt{2}}}/2}, \quad \sin(\theta) = \boxed{\sqrt{2-\sqrt{2+\sqrt{2}}}/2}.$$

(e) By the double-angle formula,

$$1 = \cos(2\theta) = 2(\cos(\theta))^2 - 1 = 1 - 2(\sin(\theta))^2.$$

Thus, we have the quadratic equations,

$$(\cos(\theta))^2 = 1, \quad (\sin(\theta))^2 = 0.$$

Since $\pi/2 < \theta \leq \pi$, $\sin(\theta)$ is nonnegative and $\cos(\theta)$ is negative. Thus the solutions are

$$\cos(\theta) = \boxed{-1}, \quad \sin(\theta) = \boxed{0}.$$

(7) In each case, for two angles α and β, the cosine and sine are given. Find cosine and sine of $\alpha+\beta$. Determine the quadrant that contains the angle $\alpha+\beta$ (measured in the usual way).

$$(a)\ \cos(\alpha)=3/5, \sin(\alpha)=4/5, \cos(\beta)=5/13, \sin(\beta)=12/13,$$

$$(b)\ \cos(\alpha)=1/\sqrt{2}, \sin(\alpha)=-1/\sqrt{2}, \cos(\beta)=1/\sqrt{2}, \sin(\beta)=1/\sqrt{2},$$

$$(c)\ \cos(\alpha)=1/\sqrt{2}, \sin(\alpha)=1/\sqrt{2}, \cos(\beta)=1/2, \sin(\beta)=\sqrt{3}/2,$$

$$(d)\ \cos(\alpha)=1/2, \sin(\alpha)=\sqrt{3}/2, \cos(\beta)=\sqrt{2+\sqrt{3}}/2, \sin(\beta)=\sqrt{2-\sqrt{3}}/2,$$

$$(e)\ \cos(\alpha)=1, \sin(\alpha)=0, \cos(\beta)=c, \sin(\beta)=s,$$

$$(f)\ \cos(\alpha)=0, \sin(\alpha)=1, \cos(\beta)=c, \sin(\beta)=s,$$

$$(g)\ \cos(\alpha)=-1, \sin(\alpha)=0, \cos(\beta)=c, \sin(\beta)=s.$$

Solution to (7) (a) By the angle addition formula,

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)=15/65-48/65=\boxed{-33/65}.$$

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)=20/65+36/65=\boxed{56/65}.$$

In particular, $\pi/2<\alpha+\beta<\pi$ (up to multiples of 2π). As a double-check, notice that

$$(\cos(\alpha+\beta))^2+(\sin(\alpha+\beta))^2=1089/4225+3136/4225=1.$$

(b) By the angle addition formula,

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)=1/2-(-1/2)=\boxed{1}.$$

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)=(-1/2)+(1/2)=\boxed{0}.$$

In particular, $\alpha+\beta=0$ (up to multiples of 2π). As a double-check, notice that

$$(\cos(\alpha+\beta))^2+(\sin(\alpha+\beta))^2=1+0=1.$$

(c) By the angle addition formula,

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)=1/2\sqrt{2}-\sqrt{3}/2\sqrt{2}=\boxed{-(\sqrt{3}-1)\sqrt{2}/4}.$$

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)=1/2\sqrt{2}+\sqrt{3}/2\sqrt{2}=\boxed{(\sqrt{3}+1)\sqrt{2}/4}.$$

In particular, $\pi/2<\alpha+\beta<\pi$ (up to multiples of 2π). As a double-check, notice that

$$(\cos(\alpha+\beta))^2+(\sin(\alpha+\beta))^2=(2-\sqrt{3})/4+(2+\sqrt{3})/4=1.$$

(d) By the angle addition formula,

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \sqrt{2+\sqrt{3}}/4 - (\sqrt{3}\sqrt{2-\sqrt{3}})/4 = \boxed{\sqrt{2-\sqrt{3}}/2}.$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = (\sqrt{3}\sqrt{2+\sqrt{3}})/4 + (\sqrt{2-\sqrt{3}})/4 = \boxed{\sqrt{2+\sqrt{3}}/2}.$$

In particular, $0 < \alpha+\beta < \pi/2$ (up to multiples of 2π). As a double-check, notice that

$$(\cos(\alpha+\beta))^2 + (\sin(\alpha+\beta))^2 = (2-\sqrt{3})/4 + (2+\sqrt{3})/4 = 1.$$

(e) By the angle addition formula,

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = c - 0 = \boxed{c}.$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = 0 + s = \boxed{s}.$$

(f) By the angle addition formula,

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = 0 - s = \boxed{-s}.$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = c + 0 = \boxed{c}.$$

(g) By the angle addition formula,

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = -c - 0 = \boxed{-c}.$$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = 0 + -s = \boxed{-s}.$$

(8) In each of the following cases, use the Pythagorean identity, the double-angle formulas, and the angle addition formulas to verify the stated identity.

$$(a)\ \cos(3\theta) = \cos(2\theta+\theta) = 4(\cos(\theta))^3 - 3\cos(\theta),\quad (b)\ \sin(3\theta) = -4(\sin(\theta))^3 + 3\sin(\theta),$$

$$(c)\ \tan(2\theta) = 2\tan(\theta)/(1-(\tan(\theta))^2),\quad (d)\ \tan(\alpha+\beta) = (\tan(\alpha)+\tan(\beta))/(1-\tan(\alpha)\tan(\beta)),$$

$$(e)\ \sec(2\theta) = (\sec(\theta))^2/(2-(\sec(\theta))^2),\quad (f)\ \csc(2\theta) = (1/2)\sec(\theta)\csc(\theta),$$

$$(g)\ \sin(2\theta)/(1+\cos(2\theta)) = \tan(\theta) = (1-\cos(2\theta))/\sin(2\theta),\quad (h)\ \sin(2\theta) = 2\tan(\theta)/(1+(\tan(\theta))^2),$$

$$(i)\ \cos(2\theta) = (1-(\tan(\theta))^2)/(1+(\tan(\theta))^2).$$

Solution to (8) (a) By the angle addition formulas,

$$\cos(2\theta+\theta) = \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta).$$

By the double-angle formulas,

$$\cos(2\theta) = 2(\cos(\theta))^2 - 1, \quad \sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Thus, we have

$$\cos(3\theta) = (2(\cos(\theta))^2 - 1)\cos(\theta) - 2(\sin(\theta))^2\cos(\theta).$$

Finally, by the Pythagorean identity,

$$(\sin(\theta))^2 = 1 - (\cos(\theta))^2.$$

Altogether, this gives,

$$\cos(3\theta) = 2(\cos(\theta))^3 - \cos(\theta) - 2(1 - (\cos(\theta))^2)\cos(\theta) = 2(\cos(\theta))^3 - \cos(\theta) - 2\cos(\theta) + 2(\cos(\theta))^3.$$

Simplifying,

$$\cos(3\theta) = \boxed{4\cos^3(\theta) - 3\cos(\theta)}.$$

(b) By the angle addition formulas,

$$\sin(2\theta + \theta) = \cos(2\theta)\sin(\theta) + \sin(2\theta)\cos(\theta).$$

By the double-angle formulas,

$$\cos(2\theta) = 1 - 2(\sin(\theta))^2, \quad \sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Thus, we have

$$\sin(3\theta) = (1 - 2(\sin(\theta))^2)\sin(\theta) + 2\sin(\theta)(\cos(\theta))^2.$$

Finally, by the Pythagorean identity,

$$(\cos(\theta))^2 = 1 - (\sin(\theta))^2.$$

Altogether, this gives,

$$\cos(3\theta) = \sin(\theta) - 2(\sin(\theta))^3 + 2(1 - (\sin(\theta))^2)\sin(\theta) = \sin(\theta) - 2(\sin(\theta))^3 + 2\sin(\theta) - 2(\sin(\theta))^3.$$

Simplifying,

$$\sin(3\theta) = \boxed{-4\sin^3(\theta) + 3\sin(\theta)}.$$

(d) By the angle addition formulas,

$$\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}.$$

Multiplying both numerator and denominator by $1/\cos(\alpha)\cos(\beta)$ gives

$$\tan(\alpha+\beta) = \frac{(\sin(\alpha)/\cos(\alpha)) + (\sin(\beta)/\cos(\beta))}{1 - (\sin(\alpha)/\cos(\alpha))(\sin(\beta)/\cos(\beta))}.$$

In other words,

$$\tan(\alpha+\beta) = \boxed{(\tan(\alpha)+\tan(\beta))/(1-\tan(\alpha)\tan(\beta))}.$$

(c) Plugging in $\beta = \alpha = \theta$ in the previous identity gives

$$\tan(\theta+\theta) = \frac{\tan(\theta)+\tan(\theta)}{1-\tan(\theta)\tan(\theta)}.$$

Simplifying,

$$\tan(2\theta) = \boxed{2\tan(\theta)/(1-\tan^2(\theta))}.$$

(e) By the double-angle formulas,

$$\frac{1}{\cos(2\theta)} = \frac{1}{2\cos^2(\theta)-1}.$$

Multiplying both numerator and denominator by $\sec^2(\theta)$ gives

$$\sec(2\theta) = \frac{\sec^2(\theta)}{2\sec^2(\theta)\cos^2(\theta)-\sec^2(\theta)}.$$

Finally, using that $\sec(\theta)\cos(\theta)$ equals 1, this gives

$$\sec(2\theta) = \boxed{\sec^2(\theta)/(2-\sec^2(\theta))}.$$

(f) By the double-angle formulas,

$$\frac{1}{\sin(2\theta)} = \frac{1}{2\sin(\theta)\cos(\theta)} = \frac{1}{2}\cdot\frac{1}{\sin(\theta)}\cdot\frac{1}{\cos(\theta)}.$$

Since $1/\sin(\theta)$ equals $\csc(\theta)$, and since $1/\cos(\theta)$ equals $\sec(\theta)$, this gives

$$\csc(2\theta) = \boxed{(1/2)\sec(\theta)\csc(\theta)}).$$

(g) By the double-angle formulas,

$$\cos(2\theta) = 2\cos^2(\theta)-1 = 1-2\sin^2(\theta).$$

Thus,

$$1+\cos(2\theta) = 2\cos^2(\theta), \quad 1-\cos(2\theta) = 2\sin^2(\theta).$$

Also by the double-angle formulas,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Thus,

$$\frac{\sin(2\theta)}{1+\cos(2\theta)} = \frac{2\sin(\theta)\cos(\theta)}{2\cos^2(\theta)} = \frac{\sin(\theta)}{\cos(\theta)}.$$

Similarly,

$$\frac{1-\cos(2\theta)}{\sin(2\theta)} = \frac{2\sin^2(\theta)}{2\sin(\theta)\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)}.$$

This gives,

$$\frac{\sin(2\theta)}{1+\cos(2\theta)} = \boxed{\tan(\theta)} = \frac{1-\cos(2\theta)}{\sin(2\theta)}.$$

(h) Beginning with the Pythagorean identity,

$$\sin^2(\theta) + \cos^2(\theta) = 1,$$

multiply both sides by $\sec^2(\theta)$ to get the identity,

$$\tan^2(\theta) + 1 = \sec^2(\theta).$$

By the double-angle formula,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Multiplying by $\sec^2(\theta)$ in the numerator and denominator gives,

$$\sin(2\theta) = \frac{2\tan(\theta)}{\sec^2(\theta)}.$$

Finally, plugging in the identity above gives

$$\sin(2\theta) = \boxed{2\tan(\theta)/(1+\tan^2(\theta))}.$$

(i) By the double-angle formula (c) above,

$$\frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\tan(\theta)}{1-\tan^2(\theta)}.$$

Thus,

$$\cos(2\theta) = \sin(2\theta)\cdot\frac{1-\tan^2(\theta)}{2\tan(\theta)}.$$

Now substituting the identity from (h),

$$\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)},$$

this gives

$$\cos(2\theta) = \boxed{(1-\tan^2(\theta))/(1+\tan^2(\theta))}.$$

(9) A right triangle has one angle θ, opposite side of length O, adjacent side of length A, and hypotenuse of length H. In each of the following cases, from the specified data, determine all of the missing data.

$$(a)\ O = 1, A = 1, H =?, \theta =?, \quad (b)\ O = 1, A = 1/\sqrt{3}, H =?, \theta =?,$$

$$(c)\ O = 2, A =?, H = 4, \theta =?, \quad (d)\ O = \sqrt{2}, A =?, H =?, \theta = \pi/4,$$

$$(e)\ O =?, A =?, H = 1, \theta = t, \quad (f)\ O =?, A = 1, H =?, \theta = t, \quad (g)\ O = 1, A =?, H =?, \theta = t.$$

Solution to (9) **(a)** By the identity,

$$\tan(\theta) = \frac{O}{A} = \frac{1}{1} = 1,$$

it follows that $\theta = \boxed{\pi/4}$. Thus,

$$H = \frac{A}{\cos(\theta)} = \frac{1}{1/\sqrt{2}} = \boxed{\sqrt{2}}.$$

(b) By the identity,

$$\tan(\theta) = \frac{O}{A} = \frac{1}{1/\sqrt{3}} = \sqrt{3},$$

it follows tht $\theta = \boxed{\pi/3}$. Thus,

$$H = \frac{O}{\sin(\theta)} = \frac{1}{\sqrt{3}/2} = \boxed{2/\sqrt{3}}.$$

(c) By the identity,

$$\sin(\theta) = \frac{O}{H} = \frac{2}{4} = 1/2,$$

it follows that $\theta = \boxed{\pi/6}$. Thus,

$$A = H\cos(\theta) = 4 \cdot \frac{\sqrt{3}}{2} = \boxed{2\sqrt{3}}.$$

(d) Since $\sin(\pi/4)$ equals $\cos(\pi/4)$ equals $1/\sqrt{2}$, also

$$H = \frac{O}{\sin(\theta)} = \frac{\sqrt{2}}{1/\sqrt{2}} = \boxed{2},$$

and

$$A = H\cos(\theta) = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}.$$

(e) Applying the right-triangle identities,

$$O = H\sin(\theta) = \boxed{\sin(t)}, \quad A = H\cos(\theta) = \boxed{\cos(t)}.$$

(f) Applying the right-triangle identities,

$$H = A/\cos(\theta) = \boxed{\sec(t)}, \quad O = A\tan(\theta) = \boxed{\tan(t)}.$$

(g) Applying the right-triangle identities,

$$H = O/\sin(\theta) = \boxed{\csc(t)}, \quad A = O\cot(\theta) = \boxed{\cot(t)}.$$

(10) Two right triangles share a common vertical side, and have adjacent horizontal sides that together sum to length B. In the first triangle, the angle opposite the vertical side has measure θ. If the hypotenuse of the first triangle has length C and the hypotenuse of the second triangle has length A, do all of the following. Find a formula for the length of the vertical side in terms of C and θ. Find a formula for the length B' of the horizontal side of the first triangle adjacent to θ in terms of C and θ. Use this formula to find the length B'' of the horizontal side of the second triangle in terms of B, C and θ. Finally, apply the Pythagorean identity to the second triangle to deduce an identity involving A, B, C and θ. This identity is the "Law of Cosines".

Solution to (10) By the right-triangle identities, the vertical side V and the horizontal side B' of the first triangle are

$$V = C\sin(\theta), \quad B' = C\cos(\theta).$$

Thus, the remainder B'' for the base B is

$$B'' = B - B' = B - C\cos(\theta).$$

From the Pythagorean identity,

$$A^2 = V^2 + (B'')^2 = (C\sin(\theta))^2 + (B - C\cos(\theta))^2 = C^2\sin^2(\theta) + (B^2 - 2BC\cos(\theta) + C^2\cos^2(\theta).$$

Since $\sin^2(\theta) + \cos^2(\theta)$ equals 1, this gives the "Law of Cosines",

$$\boxed{A^2 = B^2 - 2BC\cos(\theta) + C^2}.$$

(11) Sketch each of the graphs of $y = \sin(x)$, $y = \cos(x)$ and $y = \tan(x)$ for $-2\pi < x < 2\pi$. On each graph, indicate the portion of the graph that satisfies the horizontal line test and that is used to define the corresponding inverse trigonometric function. Next, sketch each of arcsine, arccosine and arctangent on their domain, indicating the range.

(12) Compute each of the following values without using a calculator.

$(a)\ \sin^{-1}(1/2) = \arcsin(1/2), \quad (b)\ \cos^{-1}(1/2) = \arccos(1/2), \quad (c)\ \tan^{-1}(\sqrt{3}) = \arctan(\sqrt{3}),$

$$(d)\ \sin^{-1}(0) = \arcsin(0),\quad (e)\ \cos^{-1}(0) = \arccos(0),\quad (f)\ \tan^{-1}(1) = \arctan(1),$$
$$(g)\ \sin^{-1}(-1/\sqrt{2}) = \arcsin(-1/\sqrt{2}),\ (h)\ \cos^{-1}(-1/\sqrt{2}) = \arccos(-1/\sqrt{2}),\ (i)\ \tan^{-1}(0) = \arctan(0),$$
$$(j)\ \sin^{-1}(-\sqrt{3}/2) = \arcsin(-\sqrt{3}/2),\quad (k)\ \cos^{-1}(-\sqrt{3}/2) = \arccos(-\sqrt{3}/2).$$

Solution to (12) **(a)** If $\sin(\theta)$ equals $1/2$ and $-\pi/2 \leq \theta \leq \pi/2$, then θ equals $\boxed{\pi/6}$.

(b) If $\cos(\theta)$ equals $1/2$ and $0 \leq \theta \leq \pi$, then θ equals $\boxed{\pi/3}$.

(c) If $\tan(\theta)$ equals $\sqrt{3}$ and $-\pi/2 < \theta < \pi/2$, then θ equals $\boxed{\pi/3}$.

(d) If $\sin(\theta)$ equals 0 and $-\pi/2 \leq \theta \leq \pi/2$, then θ equals $\boxed{0}$.

(e) If $\cos(\theta)$ equals 0 and $0 \leq \theta \leq \pi$, then θ equals $\boxed{\pi/2}$.

(f) If $\tan(\theta)$ equals 1 and $-\pi/2 < \theta < \pi/2$, then θ equals $\boxed{\pi/4}$.

(g) If $\sin(\theta)$ equals $-1/\sqrt{2}$ and $-\pi/2 \leq \theta \leq \pi/2$, then θ equals $\boxed{-\pi/4}$.

(h) If $\cos(\theta)$ equals $-1/\sqrt{2}$ and $0 \leq \theta \leq \pi$, then θ equals $\boxed{3\pi/4}$.

(i) If $\tan(\theta)$ equals 0 and $-\pi/2 < \theta < \pi/2$, then θ equals $\boxed{0}$.

(j) If $\sin(\theta)$ equals $-\sqrt{3}/2$ and $-\pi/2 \leq \theta \leq \pi/2$, then θ equals $\boxed{-\pi/3}$.

(k) If $\cos(\theta)$ equals $-\sqrt{3}/2$ and $0 \leq \theta \leq \pi$, then θ equals $\boxed{5\pi/6}$.

(13) In each of the following cases, solve for the given value of θ in the given interval.

$$(a)\ \sin(\theta) = 1/2,\ \pi/2 \leq \theta < \pi,\ (b)\ \cos(\theta) = 1/2,\ -\pi/2 \leq \theta < 0,\ (c)\ \tan(\theta) = \sqrt{3},\ -\pi \leq \theta < -\pi/2,$$
$$(d)\ \sin(\theta) = 0,\ -3\pi/2 < \theta < -\pi/2,\ (e)\ \cos(\theta) = 0,\ -\pi < \theta < 0,\ (f)\ \tan(\theta) = 1,\ \pi/2 < \theta < 3\pi/2,$$
$$(g)\ \sin(\theta) = -1/\sqrt{2},\ \pi/2 < \theta < 3\pi/2,\quad (h)\ \cos(\theta) = -1/\sqrt{2},\ -\pi < \theta < 0,$$
$$(i)\ \tan(\theta) = 0,\ -3\pi/2 < \theta < -\pi/2,\quad (j)\ \sin(\theta) = -\sqrt{3}/2,\ -\pi/2 < \theta < \pi/2,$$
$$(k)\ \cos(\theta) = -\sqrt{3}/2,\ 0 \leq \theta \leq \pi.$$

Solution to (13) **(a)** If $\sin(\theta)$ equals $1/2$ and $\pi/2 \leq \theta < \pi$, then θ equals $\boxed{5\pi/6}$.

(b) If $\cos(\theta)$ equals $1/2$ and $-\pi/2 \leq \theta < 0$, then θ equals $\boxed{-\pi/3}$.

(c) If $\tan(\theta)$ equals $\sqrt{3}$ and $-\pi < \theta < -\pi/2$, then θ equals $\boxed{-2\pi/3}$.

(d) If $\sin(\theta)$ equals 0 and $-3\pi/2 < \theta < -\pi/2$, then θ equals $\boxed{-\pi}$.

(e) If $\cos(\theta)$ equals 0 and $-\pi < \theta < 0$, then θ equals $\boxed{-\pi/2}$.

(f) If $\tan(\theta)$ equals 1 and $\pi/2 < \theta < 3\pi/2$, then θ equals $\boxed{5\pi/4}$.

(g) If $\sin(\theta)$ equals $-1/\sqrt{2}$ and $\pi/2 < \theta < 3\pi/2$, then θ equals $\boxed{5\pi/4}$.

(h) If $\cos(\theta)$ equals $-1/\sqrt{2}$ and $-\pi < \theta < 0$, then θ equals $\boxed{-3\pi/4}$.

(i) If $\tan(\theta)$ equals 0 and $-3\pi/2 < \theta < -\pi/2$, then θ equals $\boxed{-\pi}$.

(j) If $\sin(\theta)$ equals $-\sqrt{3}/2$ and $-\pi/2 < \theta < \pi/2$, then θ equals $\boxed{-\pi/3}$.

(k) If $\cos(\theta)$ equals $-\sqrt{3}/2$ and $0 \leq \theta \leq \pi$, then θ equals $\boxed{5\pi/6}$.

(14) In each of the following cases, find a formula for $f(x)$ that involves only addition, subtraction, multiplication, division and square roots (but no trigonometric or inverse trigonometric functions).

$(a)\ \cos(\arcsin(x)),\ (b)\ \sin(\arccos(x)),\ (c)\ \tan(\arccos(x)),\ (d)\ \tan(\arcsin(x)),\ (e)\ \cos(\arctan(x)),$

$(f)\ \sin(\arctan(x)),\ (g)\ \sin(\pi/2-\arcsin(x)),\ (h)\ \cos(\pi/2-\arccos(x)),\ (i)\ \tan(\pi/2-\arctan(x)),$

$(j)\ \sin(\pi/2-\arccos(x)),\ (k)\ \cos(\pi/2-\arcsin(x)),\ (l)\ \tan(\pi/2-\arcsin(x)).$

Solution to (14) (a) If $\sin(\theta)$ equals x, then the Pythagorean identity gives,

$$\cos^2(\theta) = 1 - \sin^2(\theta) = 1 - x^2.$$

Since $\cos(\theta)$ is nonnegative for $-\pi/2 \leq \theta \leq \pi/2$, this gives,

$$\cos(\theta) = \cos(\arcsin(x)) = \boxed{\sqrt{1-x^2}}.$$

(b) If $\cos(\theta)$ equals x, then the Pythagorean identity gives,

$$\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - x^2.$$

Since $\sin(\theta)$ is nonnegative for $0 \leq \theta \leq \pi$, this gives,

$$\sin(\theta) = \sin(\arccos(x)) = \boxed{\sqrt{1-x^2}}.$$

(c) If $\cos(\theta)$ equals x, then Pythagorean identity gives,

$$\tan^2(\theta) = \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1-x^2}{x^2}.$$

Since $\tan(\theta)$ is positive for $0 \leq \theta < \pi/2$ and since $\tan(\theta)$ is negative for $\pi/2 < \theta < \pi$, this gives

$$\tan(\theta) = \boxed{\sqrt{1-x^2}/x}.$$

(d) If $\sin(\theta)$ equals x, then the Pythagorean identity gives,

$$\tan^2(\theta) = \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{x^2}{1-x^2}.$$

Since $\tan(\theta)$ is positive for $0 \leq \theta < \pi/2$ and since $\tan(\theta)$ is negative for $-\pi/2 < \theta < 0$, this gives

$$\tan(\theta) = \boxed{x/\sqrt{1-x^2}}.$$

(e) If $\tan(\theta)$ equals x, then the Pythagorean identity gives,

$$\sec^2(\theta) = 1 + \tan^2(\theta) = 1 + x^2.$$

Since $\cos^2(\theta)$ equals $1/\sec^2(\theta)$, this gives,

$$\cos^2(\theta) = \frac{1}{1+x^2}.$$

Since $\cos(\theta)$ is nonnegative for $-\pi/2 \leq \theta \leq \pi/2$, this gives

$$\cos(\theta) = \cos(\arctan(x)) = \boxed{1/\sqrt{1+x^2}}.$$

(f) If $\tan(\theta)$ equals x, then using (e) above,

$$\sin^2(\theta) = \tan^2(\theta)\cos^2(\theta) = x^2 \cdot \frac{1}{1+x^2}.$$

Since $\sin(\theta)$ is negative for $-\pi/2 < \theta < 0$, and since $\sin(\theta)$ is positive for $0 < \theta < \pi/2$, this gives

$$\sin(\theta) = \sin(\arctan(x)) = \boxed{x/\sqrt{1+x^2}}.$$

(g) If $\sin(\theta)$ equals x, then $\sin(\pi/2-\theta)$ equals $\cos(\theta)$. Also, if $-\pi/2 \leq \theta \leq \pi/2$, then $0 \leq \pi/2-\theta \leq \pi$, so that $\sin(\pi/2-\theta)$ is nonnegative. By the Pythagorean identity, this gives,

$$\sin(\pi/2 - \arcsin(x)) = \cos(\theta) = \boxed{\sqrt{1-x^2}}.$$

(h) If $\cos(\theta)$ equals x, then $\cos(\pi/2-\theta)$ equals $\sin(\theta)$. Also, if $0 \leq \theta \leq \pi$, then $-\pi/2 \leq \pi/2-\theta \leq \pi/2$, so that $\cos(\pi/2-\theta)$ is nonnegative. By the Pythagorean identity, this gives,

$$\cos(\pi/2 - \arccos(x)) = \sin(\theta) = \boxed{\sqrt{1-x^2}}.$$

(i) If $\tan(\theta)$ equals x, then $\tan(\pi/2-\theta)$ equals $1/\tan(\theta)$. This gives,

$$\tan(\pi/2 - \arctan(x)) = \boxed{1/x}.$$

(j) Since $\sin(\pi/2 - \theta)$ equals $\cos(\theta)$, if $\cos(\theta)$ equals x, then $\sin(\pi/2 - \theta)$ equals x,

$$\sin(\pi/2 - \arccos(x)) = \boxed{x}.$$

(k) Since $\cos(\pi/2 - \theta)$ equals $\sin(\theta)$, if $\sin(\theta)$ equals x, then $\cos(\pi/2 - \theta)$ equals x,

$$\cos(\pi/2 - \arcsin(x)) = \boxed{x}.$$

(l) Since $\sin(\pi/2 - \theta)$ equals $\cos(\theta)$ and $\cos(\pi/2 - \theta)$ equals $\sin(\theta)$, also,

$$\tan(\pi/2 - \theta) = \frac{\sin(\pi/2 - \theta)}{\cos(\pi/2 - \theta)} = \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)}.$$

By (d), if $\sin(\theta)$ equals x, then

$$\tan(\theta) = \frac{x}{\sqrt{1 - x^2}}.$$

Thus, also

$$\tan(\pi/2 - \theta) = \frac{1}{\tan(\theta)} = \boxed{\sqrt{1 - x^2}/x}.$$

(15) Consider the rational functions

$$f(x) = x/x, \quad g(x) = (x+1)/(x^2 - x), \quad h(x) = 1/(x^2 - 1).$$

For each rational function, find the maximal domain where the expression is defined. **Nota Bene.** The function is **not defined** at points where it is of the form $0/0$.

Solution to (15) For $f(x) = x/x$, the maximal domain is $\boxed{(-\infty, 0) \cup (0, \infty)}$. For $g(x) = (x + 1)/(x^2 - x)$, since the denominator is zero precisely when $x = 0$ and $x = 1$, the maximal domain is $\boxed{(-\infty, 0) \cup (0, 1) \cup (1, \infty)}$. For $h(x) = 1/(x^2 - 1)$, since the denominator is zero precisely when $x = -1$ and $x = 1$, the maximal domain is $\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$.

(16) For each function from the previous problem, find the horizontal asymptote (if one exists), and find every vertical asymptote. **Nota Bene.** A point $x = a$ where the function is of the form $0/0$ may **fail** to give a vertical asymptote – it depends on the factored form of the expression.

Solution to (16) The reduced fraction that is equivalent to $f(x)$ away from $x = 0$ is 1. Thus the horizontal asymptote of $f(x)$ is $\boxed{y = 1}$. There is $\boxed{\text{no vertical asymptote}}$ of $f(x)$.

Since the degree, 2, of the denominator of $g(x)$ is greater than the degree, 1, of the numerator, $g(x)$ has a horizontal asymptote $\boxed{y = 0}$. Since the fraction is reduced, and since the denominator is zero for $x = 0$ and $x = 1$, $g(x)$ has two vertical asymptotes: $\boxed{x = 0 \text{ and } x = 1}$.

Since the degree, 2, of the denominator of $h(x)$ is greater than the degree, 0, of the numerator, $h(x)$ has a horizontal asymptote $\boxed{y = 0}$. Since the fraction is reduced, and since the denominator is zero for $x = -1$ and $x = 1$, $h(x)$ has two vertical asymptotes: $\boxed{x = -1 \text{ and } x = 1}$.

(17) Repeat the previous two exercises for each of the following

(a) $f(x)+g(x)$, *(b)* $f(x)+h(x)$, *(c)* $g(x)+h(x)$, *(d)* $f(x)\cdot g(x)$, *(e)* $g(x)\cdot h(x)$, *(f)* $h(x)/g(x)$.

Solution to (17) (a) The sum is

$$f(x)+g(x) = \frac{x}{x} + \frac{x+1}{x^2-x} = \frac{x}{x}\cdot\frac{x-1}{x-1} + \frac{x+1}{x(x-1)} = \frac{x(x-1)+(x+1)}{x(x-1)} = \frac{x^2+1}{x(x-1)}.$$

Since the denominator is zero for $x=0$ and $x=1$, the domain of $f(x)+g(x)$ is $\boxed{(-\infty,0)\cup(0,1)\cup(1,\infty)}$. Since the fraction is reduced, and since the denominator is zero for $x=0$ and $x=1$, there are two vertical asymptotes: $\boxed{x=0 \text{ and } x=1}$. Finally, from the factorization

$$f(x)+g(x) = \frac{x^2+1}{x^2-x} = \frac{x^2(1+(1/x)^2)}{x^2(1-(1/x))} = \frac{1+(1/x)^2}{1-(1/x)},$$

the limit as x approaches $+\infty$, or as x approaches $-\infty$, is

$$\lim_{x\to\pm\infty}(f(x)+g(x)) = \frac{1+0}{1-0} = 1.$$

Thus $f(x)+g(x)$ has a horizontal asymptote: $\boxed{y=1}$.

(b) The sum is

$$f(x)+h(x) = \frac{x}{x} + \frac{1}{x^2-1} = \frac{x}{x}\cdot\frac{x^2-1}{x^2-1} + \frac{1}{x^2-1}\cdot\frac{x}{x} = \frac{x(x^2-1)+x}{x(x^2-1)} = \frac{x^3}{x(x^2-1)}.$$

Since the denominator is zero for $x=0$, for $x=-1$, and for $x=1$, the domain of $f(x)+g(x)$ is $\boxed{(-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,\infty)}$. The reduced fraction that equals $f(x)+g(x)$ away from $x=-1,0,1$ is

$$f(x)+h(x) = \frac{x^2}{(x+1)(x-1)}.$$

Since this fraction is reduced, and since the denominator is zero for $x=-1$ and $x=1$, there are two vertical asymptotes: $\boxed{x=-1 \text{ and } x=1}$. Finally, from the factorization

$$f(x)+h(x) = \frac{x^2}{x^2-1} = \frac{x^2}{x^2(1-(1/x)^2)} = \frac{1}{1-(1/x)^2},$$

the limit as x approaches $+\infty$, or as x approaches $-\infty$, is

$$\lim_{x\to\pm\infty}(f(x)+g(x)) = \frac{1}{1-0} = 1.$$

Thus $f(x)+h(x)$ has a horizontal asymptote: $\boxed{y=1}$.

(c) The sum is

$$g(x) + h(x) = \frac{x+1}{x^2 - x} + \frac{1}{x^2 - 1} = \frac{x+1}{x(x-1)} \cdot \frac{x+1}{x+1} + \frac{1}{(x-1)(x+1)} \cdot \frac{x}{x} =$$

$$\frac{(x+1)^2 + x}{x(x-1)(x+1)} = \frac{(x + (3+\sqrt{5})/2)(x + (3-\sqrt{5})/2)}{x(x-1)(x+1)}.$$

Since the denominator is zero for $x = -1$, $x = 0$, and $x = 1$, the domain of $g(x) + h(x)$ is $\boxed{(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)}$. Since the fraction is reduced, and since the denominator is zero for $x = -1$, $x = 0$ and $x = 1$, there are three vertical asymptotes: $\boxed{x = -1,\ x = 0 \text{ and } x = 1}$. Finally, since the denominator has higher degree, 3, than the degree, 2, of the numerator, $g(x)+h(x)$ has a horizontal asymptote, $\boxed{y = 0}$.

(18) In each of the following cases, compute the divison with remainder of $q(x)$ into $p(x)$. Use this to find the proper fraction form of $p(x)/q(x)$.

$$(a)\ p(x) = 2x^2 + 3x + 1,\ q(x) = x,\quad (b)\ p(x) = 2x^2 + 3x + 1,\ q(x) = x + 1,$$

$$(c)\ p(x) = x^2 + x + 1,\ q(x) = x^2 - x + 1,\quad (d)\ p(x) = x^3 + x^2 + x + 1,\ q(x) = x^2 + x + 1,$$

$$(e)\ p(x) = x^2 + 2x + 3,\ q(x) = 3x - 2.$$

Solution to (18) (a) The long division is

$$p(x) = 2x^2 + 3x + 1 = a(x)q(x) + r(x) \boxed{(2x+3)q(x) + 1},$$

for the polynomial $a(x) = 2x + 3$ and for the remainder $r(x) = 1$ of degree $< \deg(q)$. Thus the proper fraction is

$$\frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)} = \boxed{(2x+3) + (1/x)}.$$

(b) The long division is

$$p(x) = 2x^2 + 3x + 1 = a(x)q(x) + r(x) \boxed{(2x+1)q(x) + 0},$$

for the polynomial $a(x) = 2x + 1$ and for the remainder $r(x) = 0$ of degree $< \deg(q)$. Thus the proper fraction is

$$\frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)} = \boxed{(2x+1) + (0/(x+1))}.$$

(c) The long division is

$$p(x) = x^2 + x + 1 = a(x)q(x) + r(x) \boxed{1q(x) + 2x},$$

for the polynomial $a(x) = 1$ and for the remainder $r(x) = 2x$ of degree $< \deg(q)$. Thus the proper fraction is

$$\frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)} = \boxed{1 + 2x/(x^2 + x + 1)}.$$

(d) The long division is

$$p(x) = x^3 + x^2 + x + 1 = a(x)q(x) + r(x)\boxed{xq(x) + 1},$$

for the polynomial $a(x) = x$ and for the remainder $r(x) = 1$ of degree $< \deg(q)$. Thus the proper fraction is

$$\frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)} = \boxed{x + 1/(x^2 + x + 1)}.$$

(e) The long division is

$$p(x) = x^2 + 2x + 3 = a(x)q(x) + r(x)\boxed{((3x + 8)/9)q(x) + (43/9)},$$

for the polynomial $a(x) = (3x + 8)/9$ and for the remainder $r(x) = 43/9$ of degree $< \deg(q)$. Thus the proper fraction is

$$\frac{p(x)}{q(x)} = a(x) + \frac{r(x)}{q(x)} = \boxed{(3x + 8)/9 + 43/(9(3x - 2))}.$$