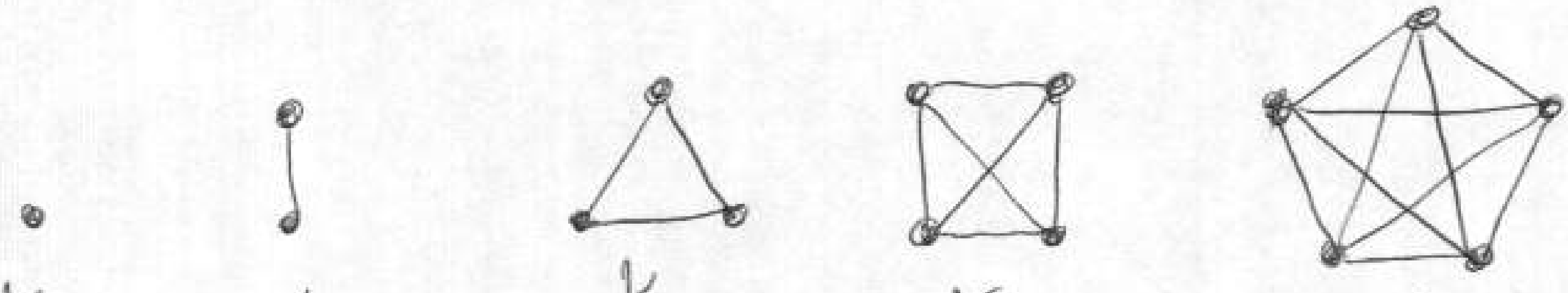
Ch. 6 (continued.) The Brute Force Algorithm

Consider the complete graph K_N of N-vertices. (We previously called this an N-clique).



K_1 K_2 K_3 K_4 K_5

We learned a few classes ago the following:

- The degree of each vertex in K_N is $N-1$.
- The # of edges in K_N is $\frac{N(N-1)}{2}$.

We can add the following to this list:

- The # Hamilton paths in K_N is $N!$
- The # Hamilton circuits in K_N is $(N-1)!$.

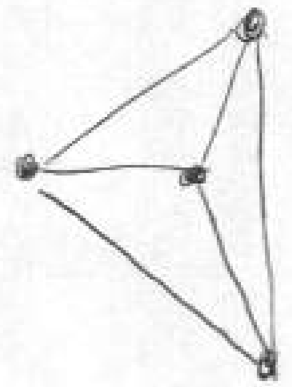
These can be explained as follows.

When creating a Hamilton path in K_N we have a certain number of choices at each step. First, we need to choose a starting vertex: there are N choices. Next, we need to choose which vertex to go to next: there are $N-1$ choices (we can't stay at our vertex). At the next step, we need to choose a vertex to proceed to that we haven't been to: there are $N-2$ choices, since we've already visited two vertices. Continuing in this fashion and multiplying all of our possible choices we get

$$\#\text{Hamilton paths} = N \times (N-1) \times (N-2) \times \cdots = N!$$

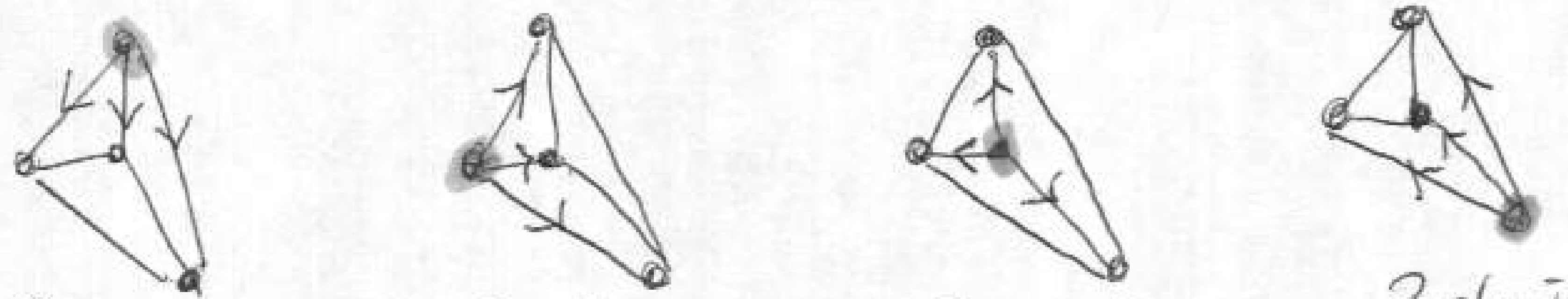
The argument for the number of Hamilton circuits is the same, except the choice of an initial vertex (the N above) is not relevant so we obtain $(N-1)!$ instead.

Here's the case of $N=4$:

K_4 

To start a Hamilton path we choose a vertex. There are 4 choices. $(N=4)$

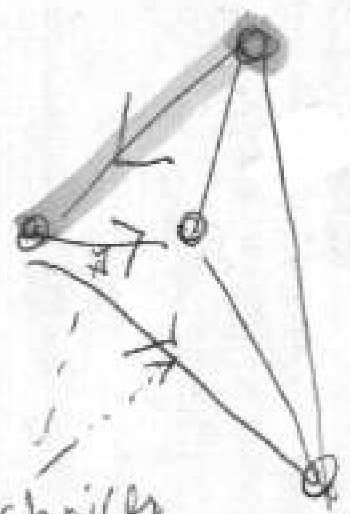
After choosing a vertex, regardless of our choice, there are 3 possible edges to proceed on to:



3 choices 3 choices 3 choices 3 choices

(here $3=N-1$)

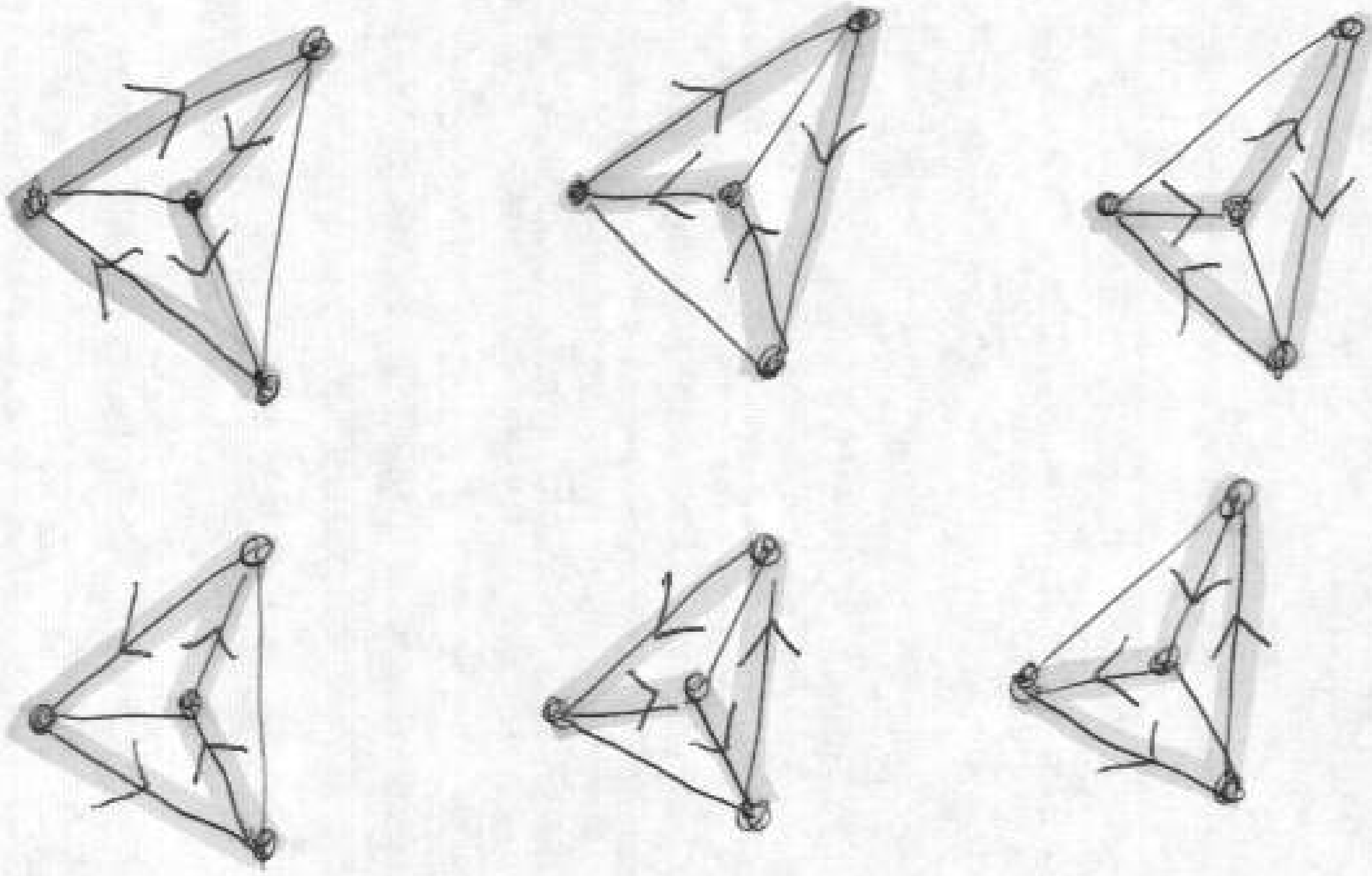
In any of the 4 cases, after making a choice of the 3 edges there are then $N-2=2$ choices to proceed. For example,



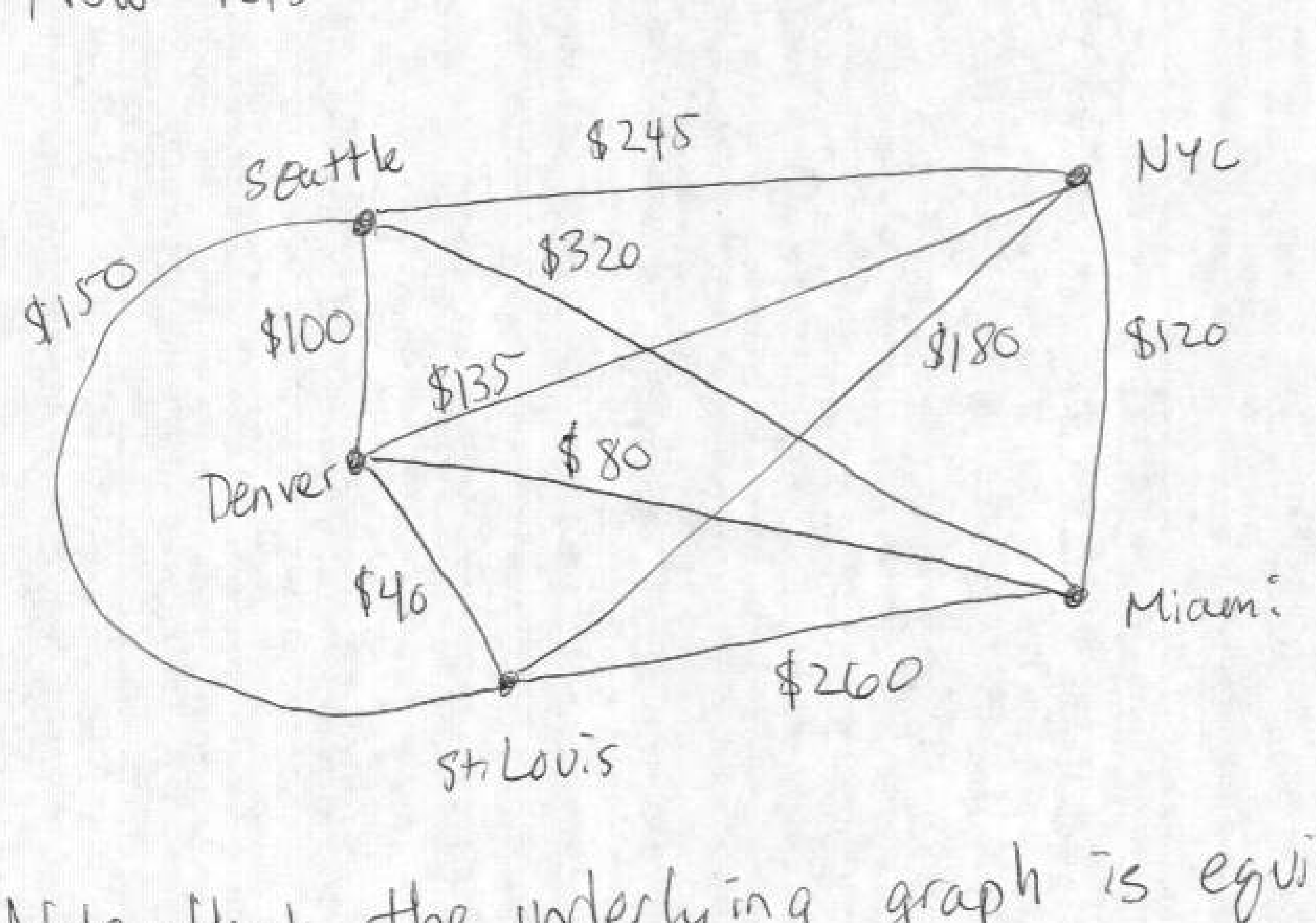
2 choices to proceed

Continuing we get $4\times3\times2\times1=4!=24$ Hamilton paths.

The result says there are $(4-1)! = 3! = 3\times 2 = 6$ Hamilton circuits for K_4. Here they are:



Now let's return to our TSP.



Note that the underlying graph is equivalent to K_5! (K_5 = )

This is an example of a <u>weighted graph</u>. A weighted graph in general is just a graph with numbers assigned to each of the edges (here these are the costs of the flights).

To find an optimal tour in our TSP we will first use the <u>Brute Force Algorithm</u>:

<u>Step 1</u>: Make a list of all Hamilton circuits of the underlying graph K_N

<u>Step 2</u>: Calculate the total weight of each Hamilton circuit.

<u>Step 3</u>: Choose a Hamilton circuit with the least total weight.

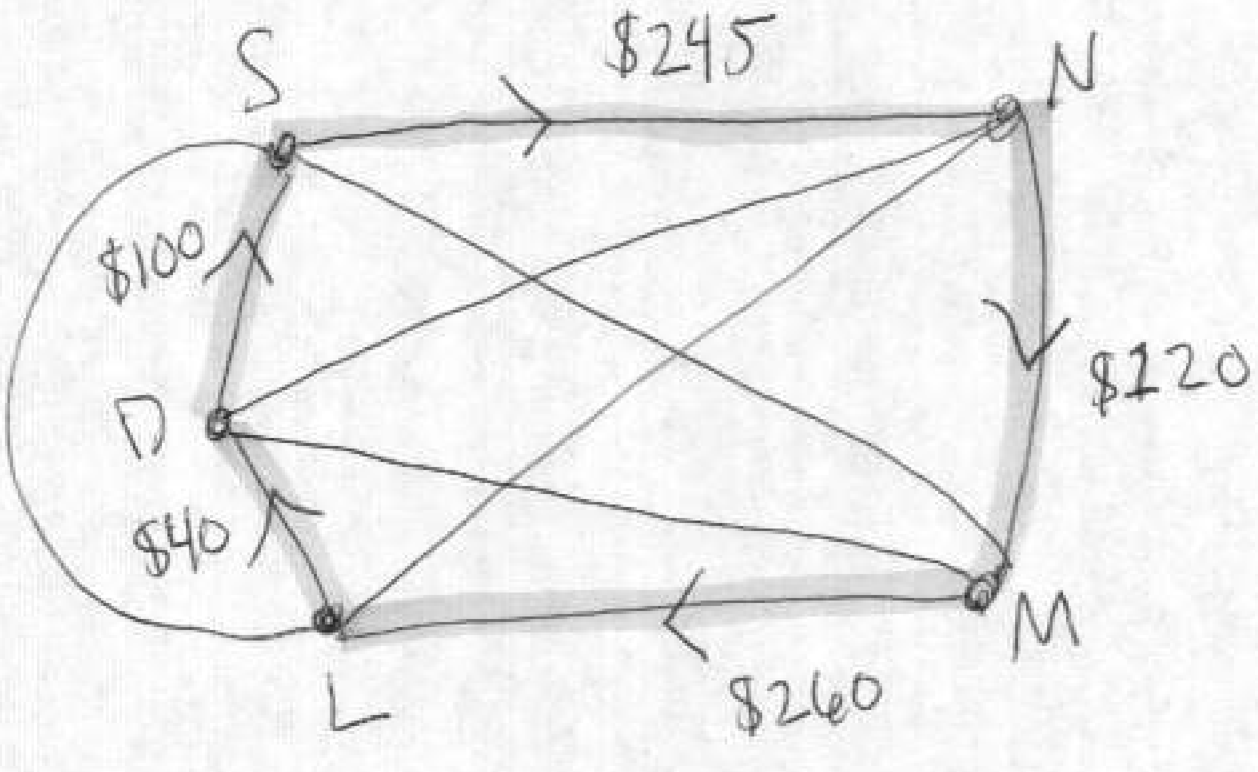
The <u>total weight</u> of a Hamilton circuit is the sum of the numbers (i.e. costs) on the edges that the circuit covers.

We apply this algorithm to our problem, which has $N=5$. For brevity we write N for NYC, S for Seattle, D for Denver, L for St. Louis, and M for Miami.

Tour	Total Cost
N, M, L, D, S, N	\$765
N, M, L, S, D, N	\$765
N, M, D, L, S, N	\$635
N, L, M, D, S, N	\$865
N, M, S, D, L, N	\$760
N, L, S, M, D, N	\$865
N, L, D, M, S, N	\$865
N, D, S, M, L, N	\$995
N, D, M, L, S, N	\$870
N, M, D, S, L, N	\$630
N, M, S, L, D, N	\$765
N, D, L, M, S, N	\$1000

(There are 12 other tours (Hamilton circuits) but they are each a reversal of some tour listed above, so its cost is recorde

For example, to compute the total cost of the tour N, M, L, D, S, N (the first in the table), we look at the corresponding Hamilton circuit:



The total cost of this tour is computed as

$$\$120 + \$260 + \$40 + \$100 + \$245 = \$765$$

as is listed in the table.

Looking at the table, the lowest possible total cost is $630, given by either N, M, D, S, L N or its reverse N, L, S, D, M, N. Here's the relevant Hamilton circuit (directions omitted):