Ch. 6: The Mathematics of Touring

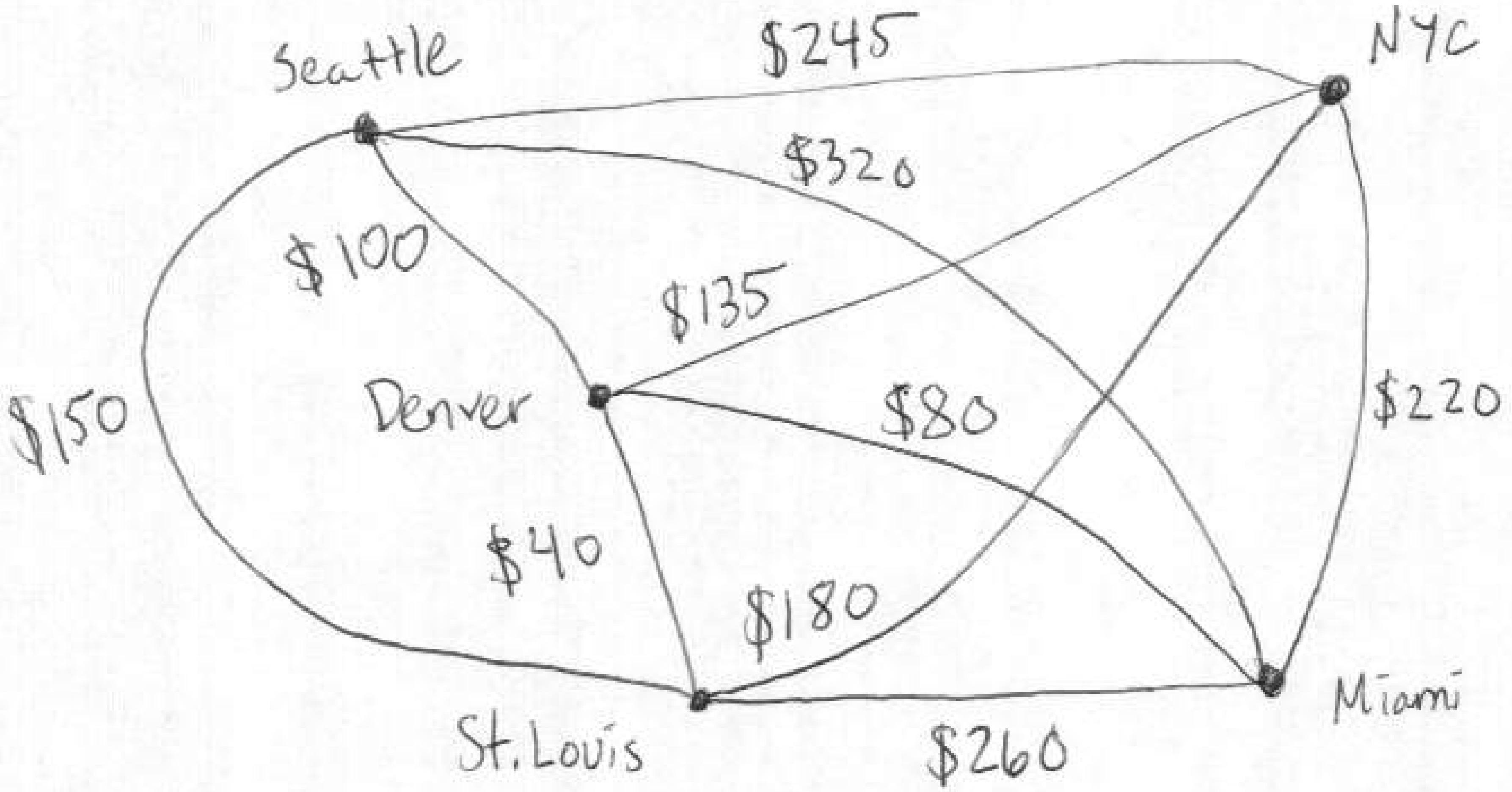
A recent college graduate (from Stony Brook!) decides to go on a trip. The main point of the trip is for the graduate to visit her friends that have scattered aross the country. The locations her friends are at include

Seattle, Denver, St. Louis & Miami.

It is her goal to visit <u>all</u> of her friends. She also wants to return to NY after her visits, so the trip should begin and end in NY.

In planning the trip, the graduate checks the airfares between each of the locations.

We can encode this data by using a graph, in which each vertex is a location, each edge indicates a path of travel between two locations, and the cost of a flight is written on each edge. Here it is:



We want a way to find a trip that not only begins and ends in NY and visits each other location, but also is the lowest cost, i.e. is <u>optimal</u> with respect to the total price of the trip.

This is a <u>travelling salesman problem</u> (TSP) which we define more formally now.

A TSP a traveller which could be a person or group of people, or more generally an object such as a bus or an unmanned rover.

The traveller must visit a set of sites. We use N to denote the total number of sites.

The set of costs are positive numbers associated with travelling from a site to another site.

"Cost" can refer to non-monetary cost, such as distance travelled or time travelled.

A solution to a TSP is a trip that starts and ends at a given site and visits all sites exactly once; such a trip is called a tour.

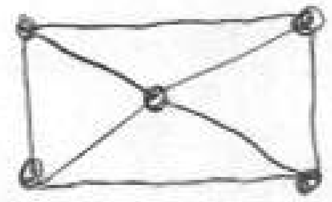
An optimal solution is a tour of minimal total cost. There is in general more than one optimal tour.

In this chapter we will study how to find optimal solutions (optimal tours) to TSP's.

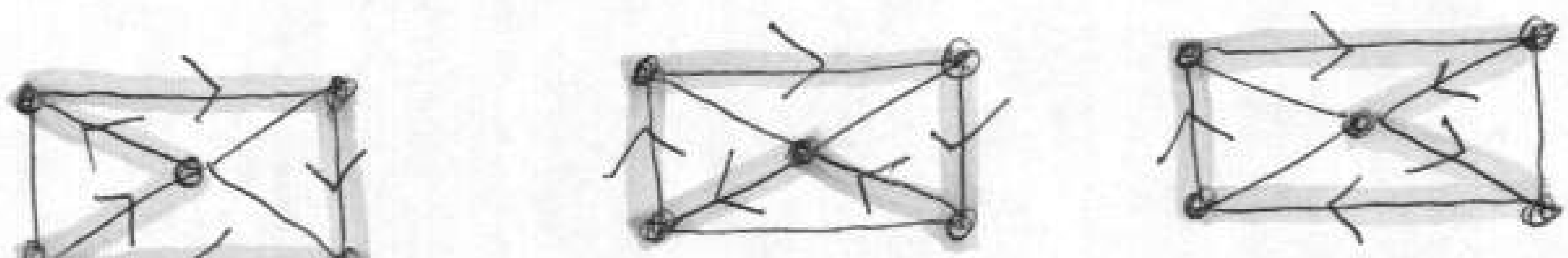
We begin by introducing some new graph theory notions related to TSP's.

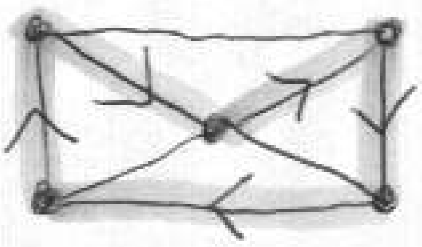
Defn. A <u>Hamilton path</u> in a connected graph is a path that visits all the vertices of the graph exactly once.

A <u>Hamilton circuit</u> in a connected graph is a circuit that visits all the vertices of the graph exactly once.

Examples: Consider the graph 

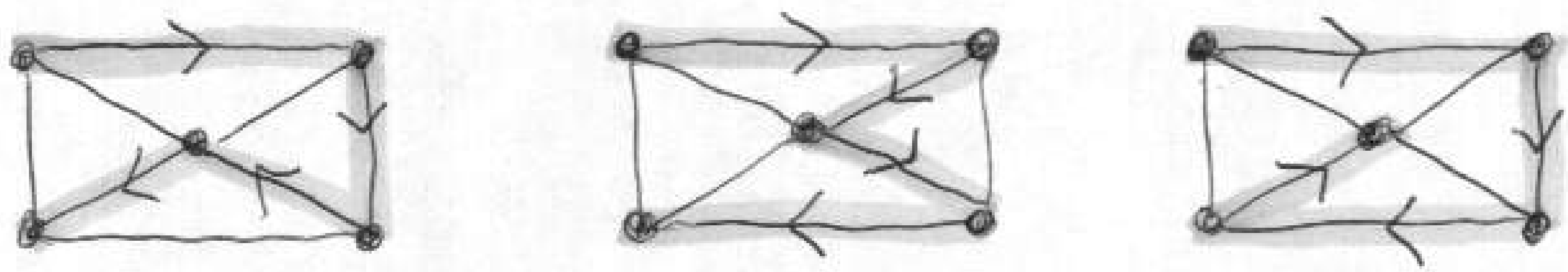
Here are some Hamilton circuits:

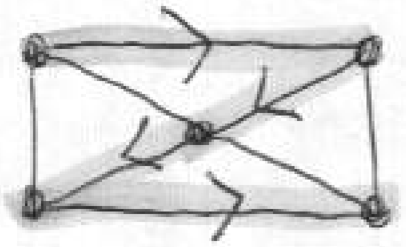




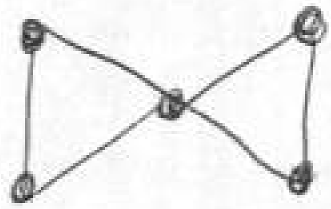
In fact there are all of them up to reversal of direction.

Here are some Hamilton paths:

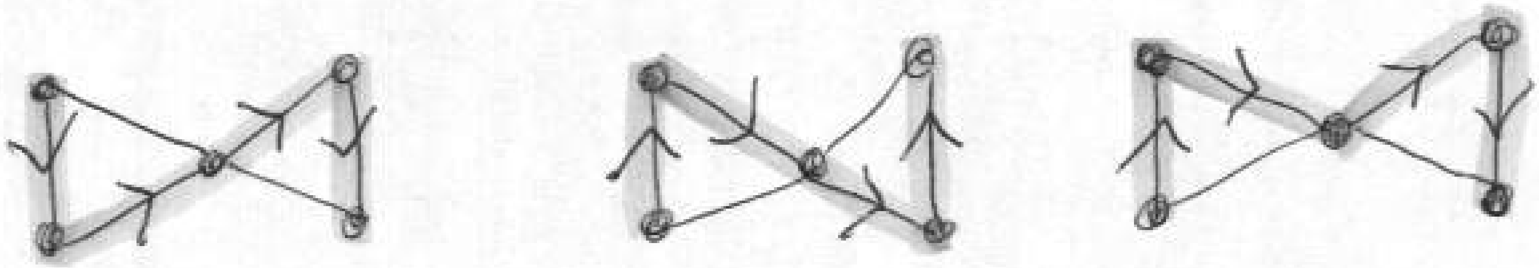


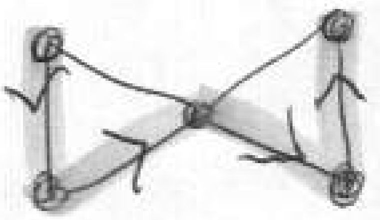


These are not all of them; it is easy to find more. (not just by reversing direction.)

Consider the graph  (the "bowtie" graph!)

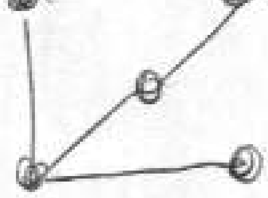
Here are some Hamilton paths:





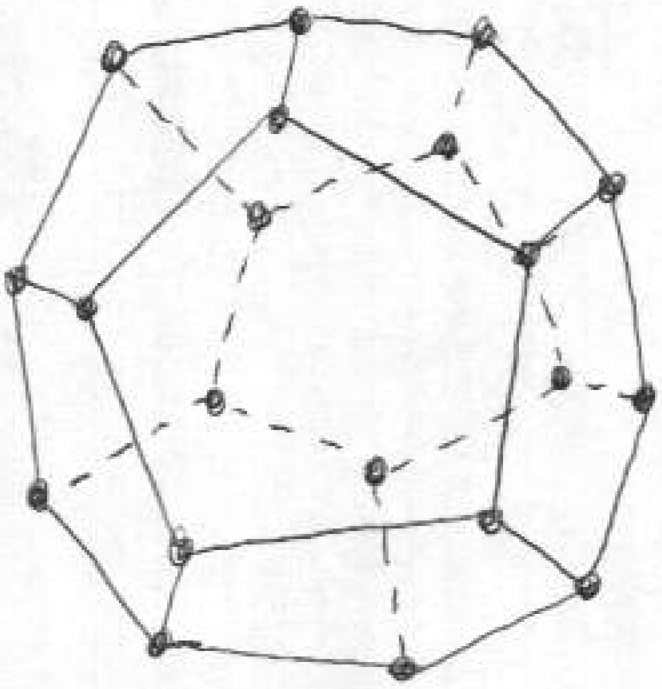
In fact, these are all of them, up to direction reversal.

However note that this graph has no Hamilton circuits: you cannot avoid hitting the middle vertex more than once when trying to create such a circuit.

Consider the graph 

This has no Hamilton circuits or Hamilton paths.

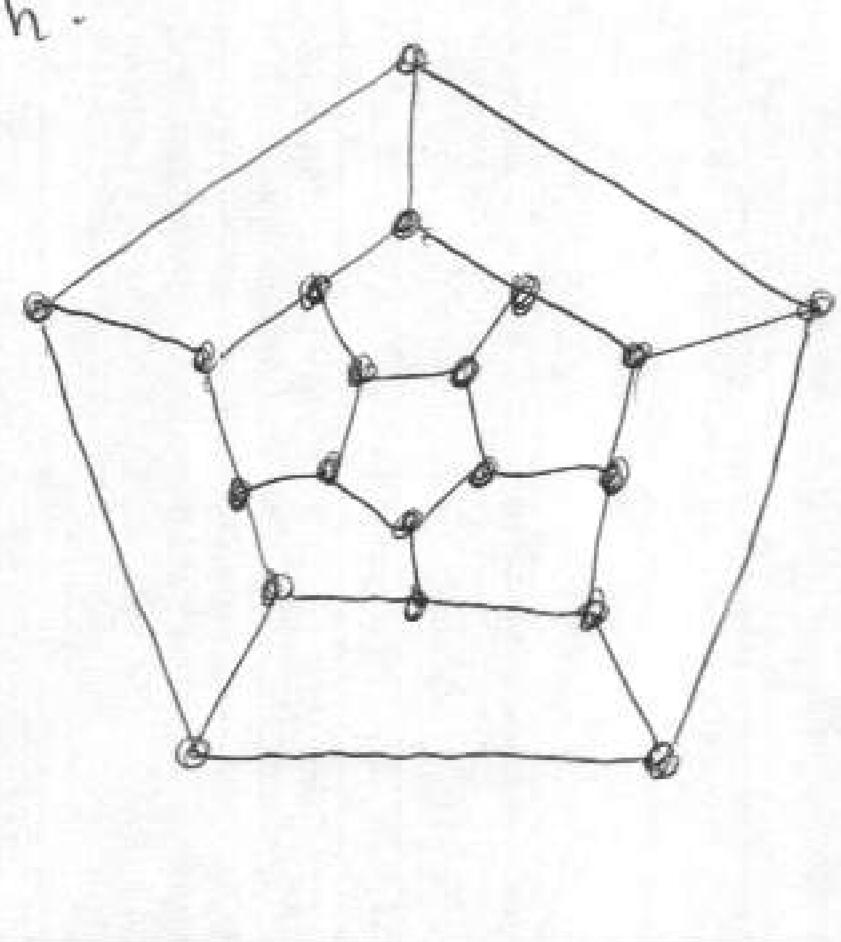
Example: consider a dodecahedron



(Look on the internet for a better illustration!)

This is a 3D object created but gluing together 12 solid pentagons along their edges.

If we only care about the vertices and edges we get a graph:



In 1857 Hamilton invented a board game which essentially has the player search for a Hamilton circuit in this graph. Here's one solution: