Sample Second Midterm Exam MAT 118

Here are some problems you can use to prepare yourself for the exam. Note that this is not an exhaustive set of problems: just because something is here doesn't mean it will be on the exam, and there may be material on the exam not represented here. You should not need a calculator to do any of these problems, but you are welcome to use one anyway.

The exam covers material since the last exam, namely chapters 9, 10 and part of 11 out of the texbook.

The exam will be held on Wednesday, November 28, during class time (9:35 AM). Do not forget to bring your student ID card or another photo ID like a driver's license.

1. If I tell you that the 29th Fibonacci number F_{29} is 514,229, and F_{31} =1,346,269, can you use that information to find F_{30}? How about F_{28}?

Solution: Since the Fibonacci numbers are defined as $F_n = F_{n-1} + F_{n-2}$, it must be true that

$$F_{31} = F_{29} + F_{30},$$

so solving for F_{30} gives

$$F_{30} = F_{31} - F_{29} = 1,346,269 - 514,229 = 832,040.$$

Similarly, $F_{30} = F_{29} + F_{28}$, so we have

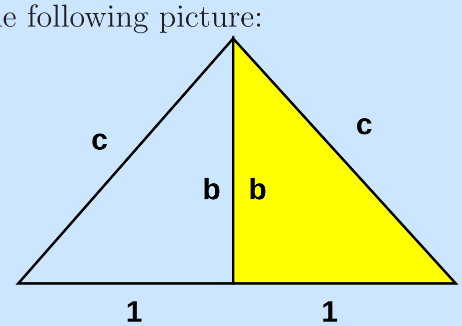
$$F_{28} = F_{30} - F_{29} = 832,040 - 514,229 = 317,811.$$

2. (a) Find a triangle with one side having length 1 that is its own gnomon. You can identify the triangle either by giving its angles or the length of the other two sides, as you like.

Solution: Since the triangle is its own gnomon, let's first figure out what it has to look like (you might skip this step and just find some triangle that works).

Since the triangle has to fit together with a copy of itself to form a new triangle, either one of the angles is 90 degrees, or two different angles add up to 180 degrees. But since the sum of all three angles in a triangle is 180 degrees, if two of them add up to 180, the other one must be zero, so that can't be. This means the triangle must be a right triangle.

That means we have the following picture:



Note that both the angles have to be 45 degrees (so that they will add up to 90 in the big triangle), and so the side marked b is actually also of length 1. Using the Pythagorean theorem, we see that $c = \sqrt{2}$.

(b) Is there more than one answer to the first part of this question (that is, is there a different triangle which is its own gnomon?) Please justify your answer fully. (An answer like "I guess not, because I couldn't find it" is not sufficient.)

Solution: No, as explained above, the triangle has to be a 45-45-90 triangle, with side lengths 1, 1, and $\sqrt{2}$. There is no other way for it to work.

3. The base of a triangle has length 5", and the base of the another is 10". If the two triangles are similar, and the area of the first is 13 square inches, what is the area of the second?

Solution: Let's let h be the height of the first triangle. Since the two triangles are similar and the base of the big one is twice that of the little one, the height of the big one must be twice that of the little one, that is, $2h$.

The area of the little triangle is 13, but we can also write it as $bh/2$. For the big triangle, the area is $(2b)(2h)/2$, that is $4bh/2$. Since $13 = bh/2$, $4bh/2 = 4 \cdot 13 = 52$. So the area of the big triangle is 52.

(If it makes you happier, you can solve for the height h of the little triangle: $13 = 5h/2$, so $h = 26/5 = 5.2$. This means the height of the big triangle is 10.4, so the area of the big triangle is $10 \cdot 10.4/2 = 104/2 = 52$. They are really the same thing.

4. Suppose β is a number such that $\beta^2 = \beta + 3$. Find two whole numbers a and b so that

$$\beta^5 = a\beta + b.$$

(It is not at all necessary to know that $\beta = (1+\sqrt{13})/2 \approx 2.3027756$ to do this problem, nor is a calculator needed.)

Solution: We have:

$$\begin{aligned}
\beta^5 &= \beta^4\beta = (\beta^2)^2\beta = (\beta+3)^2\beta \\
&= (\beta^2 + 6\beta + 9)\beta = \big((\beta+3) + 6\beta + 9\big)\beta \\
&= (7\beta + 12)\beta = 7\beta^2 + 12\beta \\
&= 7(\beta+3) + 12\beta = 7\beta + 21 + 12\beta \\
&= 19\beta + 21.
\end{aligned}$$

So $a = 19$ and $b = 21$.

5. A nuclear power plant produces 12 pounds of radioactive waste every month, which must be stored in a special tank which can hold 500 pounds of the glowing goo. On January 1 2006, there were 25 pounds of waste in the tank (I guess it came with one pound for good luck). Let P_N represent the amount of radioactive waste in the tank after N months.

   (a) Write an expression (either explicit or recursive) for P_N.

   Solution: Since we add 12 pounds each month and started with 25 pounds, we have the following recursive expression:

   $$P_N = P_{N-1} + 12 \qquad P_0 = 25$$

   Alternatively, we have

   $$P_N = 25 + 12N$$

   (b) When will the tank be full?

   Solution: We need to know for which N we have $P_N \geq 500$. If you hate using equations, you can just count the terms in the sequence

   $$25, 37, 49, 61, 73, 85, 97, 109, \cdots$$

   until it gets bigger than 500. But it is much better to just solve:

   $$500 = 25 + 12N \qquad \text{so} \qquad N = 475/12 \approx 39.583$$

   So the tank fills about halfway through the 39th month.

6. In 2000, there were 100,000 cases of equine flu reported. Each year, the number of cases decreases by 20%. Let E_n be the number of new cases of equine flu reported in the year $2000 + n$.

   (a) Write an expression (either explicit or recursive) for E_n.

   Solution: Since the number of cases decreases by 20% each year, each year the number of cases is 80% of the previous one. This means that recursively, we have

   $$E_n = 0.8E_{n-1} \qquad E_0 = 100,000$$

   The explicit equation is $E_n = 100,000 \cdot (0.8)^n$

   (b) Assuming the trend continues, how many cases of equine flu should be expected in 2015?

   Solution: We could use the recursive equation to see that the populations for 2000–2015 are

   $$10000, 8000, 6400, 5120, 4096, 32768, 2621.44, 2097.15, 1677.72,$$

   $$1342.17, 1073.74, 858.993, 687.19, 549.75, 439.80, 351.84$$

This is tedious, so more efficient would just be to calculate

$$E_{15} = 100000(0.8)^{15} \approx 351$$

or so.

7. A population of rabbits on an island covered with delicious green grass and no predators grows according to the logistic model

$$P_{n+1} = 3P_n(1 - P_n),$$

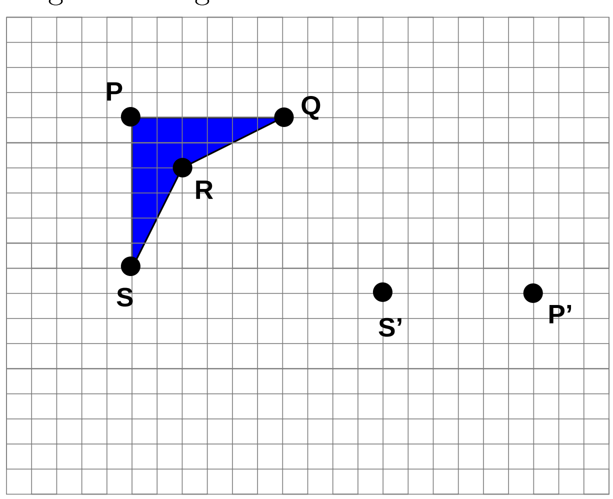
where P_n represents the number of rabbits after n years as a fraction of the island's carrying capacity. If $P_0 = 1/3$, find the population after 5 years.

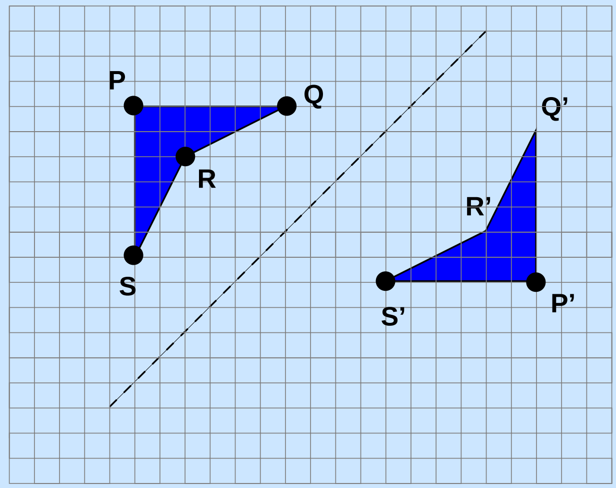
Solution: We have

$$\begin{aligned} P_0 &= \tfrac{1}{3} \\ P_1 &= 3(\tfrac{1}{3})(1 - \tfrac{1}{3}) = 3(\tfrac{1}{3})(\tfrac{2}{3}) = \tfrac{2}{3} \\ P_2 &= 3(\tfrac{2}{3})(1 - \tfrac{2}{3}) = 3(\tfrac{2}{3})(\tfrac{1}{3}) = \tfrac{2}{3} \\ P_3 &= 3(\tfrac{2}{3})(1 - \tfrac{2}{3}) = 3(\tfrac{2}{3})(\tfrac{1}{3}) = \tfrac{2}{3} \end{aligned}$$

The population stays at 2/3 from then on.

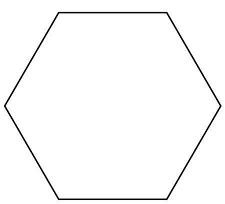
8. In the figure below, a glide reflection takes the point marked **P** to the point marked **P'** and the point marked **S** to the point marked **S'**. On the figure, indicate the axis of the reflection, and the image of the figure.





Solution:

9. How many distict symmetries does a regular hexagon have?



Solution: There are 12.

There are six rotations: by 60, 120, 180, 240, 300, and 360 degrees, and there are six reflections, indicated by the six lines in the figure below.