

# MAT360

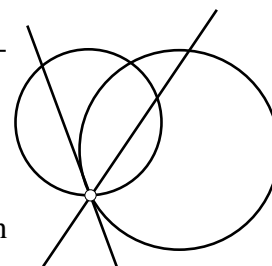
# Solutions to Midterm

1. Consider the following interpretation of the terms “point”, “line”, and “incidence”:

- A **point** is any pair of real numbers  $(x, y)$  such that  $x$  and  $y$  are not both zero. That is, any point in  $\mathbb{R}^2$  except the origin.
- A **line** is the set of **points**  $(x, y)$  as above for which there are real numbers  $a$  and  $b$  not both zero such that either

$$(x - a)^2 + (y - b)^2 = (a^2 + b^2) \quad \text{or} \quad ax = by.$$

That is, a **line** is any circle or straight line passing through the origin in  $\mathbb{R}^2$ .



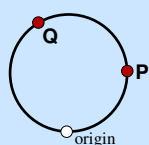
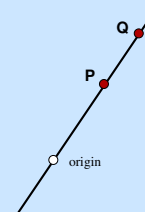
- A **point** is **incident** with a **line** if it satisfies the corresponding equation.

(a) 6 points Does this define an incidence geometry? That is, do axioms **I1**, **I2**, and **I3** hold? Fully justify your answer.<sup>1</sup>

**Solution:** Throughout, we will use **line** or **point** to refer to the objects as defined in the model, and line, point, or circle to refer to the “usual” Euclidean meanings in  $\mathbb{R}^2$ .

First, let’s check **I1**: Given two **points**  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$ , we need to confirm that they determine a unique **line**. There are two cases:

First, suppose that  $P$  and  $Q$  are collinear with the origin in  $\mathbb{R}^2$ . Either the line  $\overleftrightarrow{PQ}$  is vertical, in which case  $y_1 = y_2 = 0$  and so  $P$  and  $Q$  lie on the line  $y = 0$  (that is,  $a = 0$  and  $b = 1$ ), or  $\overleftrightarrow{PQ}$  has slope  $m$ , in which case the line is  $y = mx$  (that is,  $a = m$  and  $b = 1$ ). Thus, if  $\overleftrightarrow{PQ}$  is of line type, it is not possible to find a circle in  $\mathbb{R}^2$  passing through  $P$ ,  $Q$  and the origin.



In the other situation,  $P$ ,  $Q$ , and the origin are not collinear in  $\mathbb{R}^2$ , so there is a unique circle containing all three (see footnote). In either case, there is a unique **line** defined by  $P$  and  $Q$ .

The next two are easy. Axiom **I2** holds, since every line has infinitely many points on it. Axiom **I3** also holds easily. For example, the **points**  $(1, 0)$ ,  $(-1, 0)$ , and  $(0, 1)$  do not lie on the same **line**.

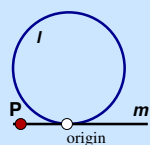
Since axioms **I1**, **I2**, and **I3** hold, this does define an incidence geometry. In fact, with suitable definitions of **between** and **congruence**, all of the other axioms except the continuity and parallel axioms hold.

<sup>1</sup>It might be useful for you to remember that for any three distinct non-collinear points in  $\mathbb{R}^2$ , there is a unique circle on which they all lie. Also, C.S. Lewis once said the following line: “The next best thing to being wise oneself is to live in a circle of those that are.”

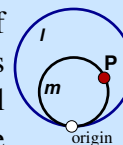
- (b) 4 points Does this model define a Euclidian, elliptic, or hyperbolic geometry (or none of the those)? Again, fully justify your answer.

**Solution:**

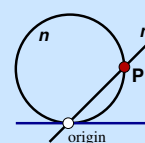
Given a **line**  $l$  and a **point**  $P$  not on  $l$ , we must determine how many **lines** passing through  $P$  which are parallel to  $l$ .



If  $l$  is of circle type, then there are two possibilities. First, if  $P$  lies on the line  $m$  which is tangent to  $l$  at the origin, **lines**  $l$  and  $m$  are parallel. There can be no other parallel **line** (all other circles or lines passing through  $P$  will intersect  $l$ ). In the other possibility, there is a unique circle through  $P$  which will be tangent to  $l$  at the origin. In both cases, there is a unique **line** through  $P$  parallel to  $l$ .



In case  $l$  is of line type, then the line in  $\mathbb{R}^2$  connecting  $P$  to the origin is a **line**  $m$  parallel to  $l$ . But also the circle which passes through  $P$  and is tangent to  $l$  at the origin is another **line**  $n$  which is parallel to  $l$ . Thus, there are two **lines** parallel to  $l$  through  $P$ .



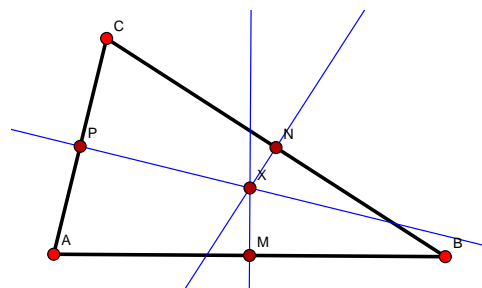
Since we sometimes have two parallels and sometimes have a unique one, the geometry is not hyperbolic, Euclidean, or elliptic.

2. 10 points Rewrite the following proof, supplying valid reasons for each of the steps.

In any triangle  $\triangle ABC$ , if the perpendicular bisectors of two of the sides of  $\triangle ABC$  meet, all three perpendicular bisectors of the three sides meet in a common point  $X$ , called the *circumcenter*.

**Proof:** Let  $M$ ,  $N$ , and  $P$  be the midpoints of sides  $\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{CA}$  respectively. Let the perpendicular bisector of  $\overline{AB}$  meet that of  $\overline{BC}$  at a point  $X$ . We must show that  $\overline{PX}$  is perpendicular to  $\overline{AC}$ .

Now,  $\triangle MXB \cong \triangle MXA$  and  $\triangle NXB \cong \triangle NXC$ . Thus  $\overline{CX} \cong \overline{AX}$ , and so  $\triangle CPX \cong \triangle APX$ . Therefore  $\overline{PX} \perp \overline{AC}$ .



**Solution:** Let  $M$ ,  $N$ , and  $P$  be the midpoints of sides  $\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{CA}$  respectively. Such points exist because we have shown that every segment has a unique midpoint.

Erect the perpendicular to  $\overline{AB}$  at  $M$ , and the perpendicular to  $\overline{BC}$  at  $N$ . Such lines exist as a consequence of axiom **C4**. Let these meet at a point  $X$  (we have taken that two of the bisectors meet as a hypothesis). We must show that  $\overline{PX}$  is the perpendicular bisector of  $\overline{AC}$ ; that is, that  $\overline{PX} \perp \overline{AC}$ .

Since  $\overline{MX} \perp \overline{AB}$ , angles  $\angle AMX$  and  $\angle BMX$  are both right, and consequently are congruent. Furthermore, by the definition of midpoint,  $\overline{AM} \cong \overline{BM}$ , and  $\overline{MX}$  is congruent to itself. Thus we can apply **SAS**, to get  $\triangle MXB \cong \triangle MXA$ . Consequently  $\overline{AX} \cong \overline{BX}$ .

By similar reasoning, we also have  $\triangle NXB \cong \triangle NXC$  and thus  $\overline{BX} \cong \overline{CX}$ .

By transitivity of congruence (axiom **C2**), we have  $\overline{CX} \cong \overline{AX}$ . Since we also have  $\overline{CP} \cong \overline{AP}$  (definition of midpoint) and  $\overline{PX} \cong \overline{PX}$ , we can apply **SSS** to see that  $\triangle CPX \cong \triangle APX$ .

Thus angles  $\angle CPX$  and  $\angle APX$  are congruent. Since they are also supplementary angles, we have shown that they must be right angles.

Therefore  $\overleftrightarrow{PX} \perp \overline{AC}$ , as desired.

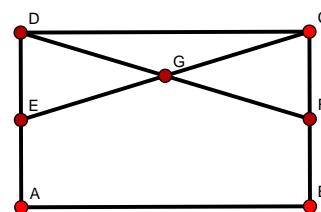
3. 10 points Let  $\square ABCD$  be a Saccheri quadrilateral, with right angles at  $A$  and  $B$ , and  $\overline{AD} \cong \overline{BC}$  as usual. Also, let  $E$  and  $F$  be the midpoints of  $\overline{AD}$  and  $\overline{BC}$  respectively, and let  $G$  be the point of intersection of  $\overline{EC}$  and  $\overline{DF}$ . Prove that  $\overline{EG} \cong \overline{FG}$ .

**Solution:** Because  $\square ABCD$  is Saccheri, we know that the summit angles  $\angle EDC$  and  $\angle FCD$  are congruent.

Furthermore, since  $E$  and  $F$  are midpoints of congruent segments, we have  $\overline{ED} \cong \overline{FC}$ . Observing that  $\overline{CD}$  is congruent to itself, we can apply **SAS** to see that triangles  $\triangle EDC$  and  $\triangle FCD$  are congruent.

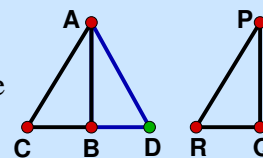
This tells us that  $\angle CDF \cong \angle ECD$ . Now we use the congruence of the summit angles again, together with angle subtraction, to conclude that  $\angle EDG \cong \angle FCG$ .

Next, notice that  $\angle EGD$  and  $\angle FGC$  are vertical angles, and hence are congruent. This, together with the previous result and the fact that  $\overline{ED} \cong \overline{FC}$ , enables us to use **AAS** to see that  $\triangle EGD \cong \triangle FGC$ . Consequently,  $\overline{EG} \cong \overline{FG}$  as desired.



4. 10 points Prove the Hypotenuse-Leg congruence condition. That is, suppose  $\triangle ABC$  and  $\triangle PQR$  are right triangles with right angles at  $\angle B$  and  $\angle Q$ . Furthermore, suppose that  $\overline{AB} \cong \overline{PQ}$  and  $\overline{AC} \cong \overline{PR}$ . Show that  $\triangle ABC \cong \triangle PQR$ . (Hint: an isosceles triangle could be helpful.)

**Solution:** Find a point  $D$  on  $\overleftrightarrow{BC}$  so that  $C \star B \star D$  with  $\overline{QR} \cong \overline{BD}$ . Since  $\overline{BD} \cong \overline{QR}$ ,  $\angle B \cong \angle Q$ , and  $\overline{AB} \cong \overline{PQ}$ , we have  $\triangle ABD \cong \triangle PQR$  by **SAS**. Thus  $\overline{AD} \cong \overline{PR} \cong \overline{AC}$ .



Therefore  $\triangle ADC$  is isosceles. Since the base angles of an isosceles triangle are congruent, we have  $\angle C \cong \angle D \cong \angle R$ .

Now we have  $\overline{AC} \cong \overline{PR}$ ,  $\angle C \cong \angle R$ , and  $\overline{BC} \cong \overline{QR}$ , we have  $\triangle ABC \cong \triangle PQR$  by **SAS**.

5. 10 points Prove that the Euclidean parallel postulate holds if and only if the following statement holds:

Let lines  $l$  and  $m$  be parallel, and let line  $t$  be perpendicular to  $l$ . Then  $t$  is also perpendicular to  $m$ .

**Solution:** Let's call the property that  $l \parallel m$  and  $t \perp l \implies t \perp m$  by the name "property S".

First, let's show that Euclidean geometry implies property S holds.

Assume  $\overleftrightarrow{l} \parallel \overleftrightarrow{m}$  and  $t \perp l$ . Now, since the geometry is Euclidean,  $t$  must intersect  $m$ , since if not, both lines  $t$  and  $l$  would be parallel to  $m$ . But since in Euclidean geometry the converse to the alternate interior angle theorem holds, the alternate interior angles formed by  $l$ ,  $LineLt$ , and  $LineLm$  must be equal. Consequently  $t \perp m$ . So property S holds.

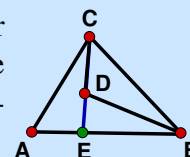
Now we show that if property S holds, the geometry must be Euclidean. One way to do this is suppose that we have a pair of parallel lines  $l$  and  $m$ . Pick two points  $P$  and  $Q$  on  $l$ , and erect perpendiculars  $t$  and  $r$  at these points. By property S,  $t \perp m$  and  $r \perp m$ . But since the resulting quadrilateral has four right angles, it is a rectangle. If rectangles exist, the geometry is Euclidean.

There are, of course, many other ways to prove both of these implications.

6. 10 points Let  $D$  be any point interior to triangle  $\triangle ABC$ . Prove that the measure of  $\angle BAC$  is less than the measure of  $\angle BDC$ .

**Solution:**

Extend  $\overline{CD}$  to a point  $E$  on  $\overline{AB}$ . We have  $A \star E \star B$  by the crossbar theorem, and so angle  $\angle DEB$  is exterior to  $\triangle ACE$ . Thus, by the exterior angle theorem,  $\angle BAC < \angle DEB$ . Also,  $\angle BDC$  is exterior to triangle  $\triangle BED$ , so we have  $\angle BAC < \angle BED < \angle BDC$ .



Most people instead constructed  $\overrightarrow{AD}$ , meeting  $\overline{BC}$  at a point  $F$ . Then they applied the exterior angle theorem to see that  $\angle CAD < \angle CDF$  and  $\angle BAD < \angle BDF$ . Finally, they used angle addition to get the desired conclusion:

$$\angle BAC = \angle CAD + \angle BAD < \angle CDF + \angle BDF = \angle BDC$$

That works, too.