Class Notes: March 19th, 2009

Part I: Why is there no text for this class? (Just in case you were wondering.)

Textbooks that are usually used for a course like this generally fall into two categories:

1. Many are too low level (ie. they tell you what you need to know but not why it works or why you need to use it)

2. Many focus more on how to teach, giving you really broad, general issues that one encounters in teaching.

Neither of these options fit into the philosophy of the course. Our focus shouldn't be in amassing algorithms but rather making educated choices about when to use what algorithms and why we would use them. In addition, a major goal of this course is to try to make connections between what you learn as an undergraduate and how it relates to the secondary curriculum.

Part II: A quick reminder about probability

Let's start off with some easy probability (ie: discrete probability).

Example 1: Draw a card out of a standard deck. What is the probability that it is red?

The probability is $\dfrac{\text{the number of red cards}}{\text{the total number of cards}} = \dfrac{26}{52} = \dfrac{1}{2}$

Example 2: Let's try the same problem but now with a deck of cards including Jokers (of which there are 2). Now our total number of cards is 54.

The probability is $\dfrac{\text{the number of red cards}}{\text{the total number of cards}} = \dfrac{26}{54} = \dfrac{13}{27}$

Now let's talk about probability with independent events (or trial). They key is that if our events/trials are independent then we multiply the individual probabilities.

Example 3: If I flip 3 coins, what is the probability that they're all heads?

The probability that any one coin will come up heads is $\frac{1}{2}$. So the probability that all the coins will come up heads is:

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

Note that we could list all possible solutions to this problem, see there are eight possible and that only one gives all heads, but listing is only feasible for small numbers. It wouldn't always work out so easily.

What if our events are not independent?

Example 4: What is the probability that we draw 2 red cards from a standard deck?

Again, since the probability of one event is $\frac{\text{the number of red cards}}{\text{the total number of cards}}$ and since we multiply probabilities of independent events, we just need account for the fact that there will be one less red card in our deck when we calculate the probability of pulling the second red card out. After taking this into account, we can treat this as two independent events, viewing the second draw as just choosing a card from a smaller deck. The resulting probability is

$$\left(\frac{\text{number of red cards}}{\text{total number of cards}}\right)\left(\frac{\text{number of red cards} - 1}{\text{total number of cards} - 1}\right) = \left(\frac{26}{52}\right)\left(\frac{25}{51}\right)$$

What if the colors we wanted were different?

Example 5: What is the probability that we draw one red card and one black card from a standard deck?

Here we have to work a little harder, since what happens in the first event determines what we want the second event to be. Let's look at it two different ways.

We can calculate $P\,(\text{one red, one black}) + P\,(\text{one black, one red})$. This is

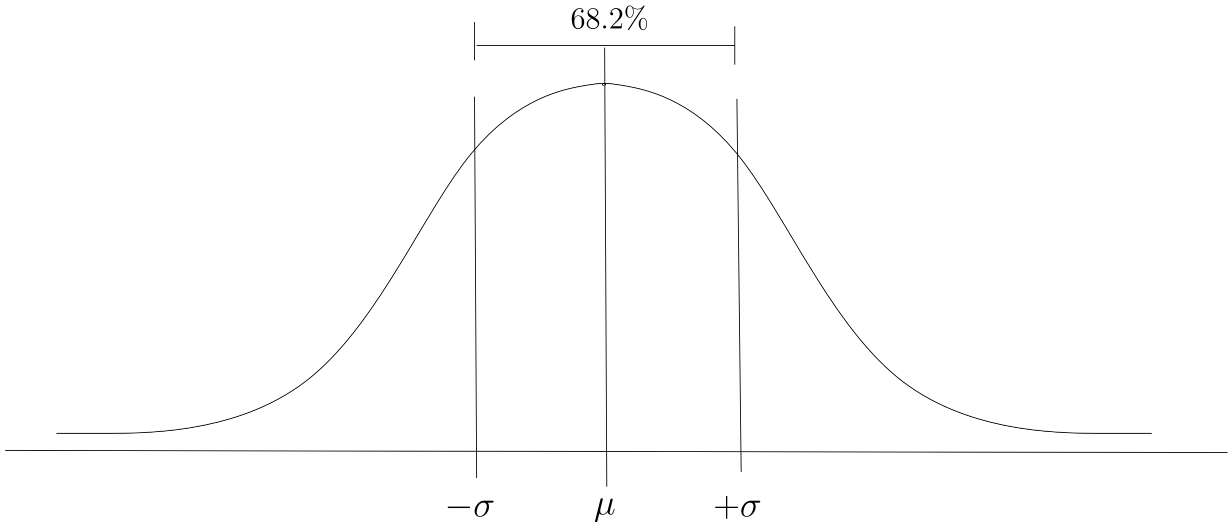
$$\left(\frac{26}{52}\right)\left(\frac{26}{51}\right) + \left(\frac{26}{52}\right)\left(\frac{26}{51}\right) = 2\left(\frac{26}{52}\right)\left(\frac{26}{51}\right) = \frac{26}{51}.$$

Alternatively, we can calculate the probability of not pulling out one red and one black. In other words, we would calculate $1 - [P\,(\text{two blacks}) + P\,(\text{two reds})]$ as

$$1 - \left[\left(\frac{26}{52}\right)\left(\frac{25}{51}\right) + \left(\frac{26}{52}\right)\left(\frac{25}{51}\right)\right] = 1 - \frac{25}{51} = \frac{26}{51}.$$

Note: One thing that is often confusing is that sometimes we discuss probability, and sometimes we discuss "the odds" of something happening. These describe the same concept, but in different ways, and so this confuses many students (and teachers!). If we say that the odds of an event occurring are 3 to 5, what we mean is that there are 3 ways it can happen and 5 ways it can fail. Thus, there are a total of $3 + 5 = 8$ ways the event could turn out. So the probability of the event happening would be $\frac{3}{8}$. It is important to keep such potential points of confusion in mind so that we can point them out and avoid them.

If we have a large number of trials then we can say there are ∞-many events. This leads to probability distributions. We can think of these distributions as a function over the sample space so that the total area under the curve is equal to 1. This represents all possible outcomes. Then calculating the probability just corresponds to finding the area under the curve corresponding to the events we are concerned with.



In the high-school curriculum, all distributions are normal. The midpoint is the mean (usually denoted by μ) and 68.2% of the data will fall within ± 1 standard deviation (σ) from the mean.

All problems on the Regents exam essentially come down to a linear change of variables to adjust what the mean and standard deviation are in each situation, and then picking off what you are interested in from the chart.

If a random sample (size n) of values x_i is taken from the collection is large enough then

$$\frac{\sum x_i}{\mathrm{n}} \longrightarrow \mu \text{ as } n \text{ gets large.}$$

And the standard deviation, denoted σ, is equal to the square root of the variance:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}$$

Part III: Enough of that, let's talk about Trig!

One of the major themes of geometry in secondary school is the idea of congruent triangles.

Definition: Two triangles are congruent of corresponding sides math up in terms of length and corresponding angles match up in terms of angle measure.

We can think of two triangles as congruent if we can get from one to the other using rigid motions in the plane, that is we can flip/translate/rotate our way from one triangle to the other.

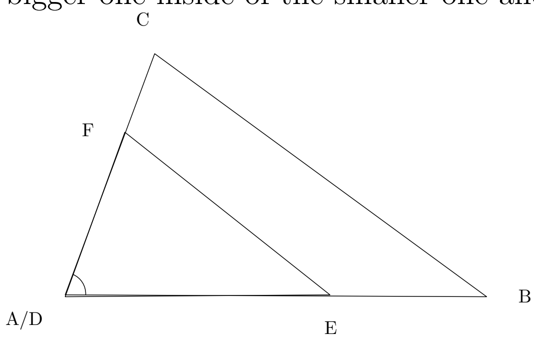
We have the usual ways to determine if two triangles are congruent:

- Side Angle Side (SAS)
- Side Side Side (SSS)
- Angle Side Angle (ASA)
- Angle Angle Side (AAS)

If we have Side Side Angle (SSA), this does not guarantee us congruent triangles. It will match one half of cases (because there we could construct two possible triangles that have the same two sides and angle).

If we have Angle Angle Angle (AAA), this leads to the idea of similar triangles (ie: triangles in which the edge lengths are in proportion). Note, we really only need two angle measures to get AAA because a triangle's interior angles must sum to 180° in Euclidean geometry.

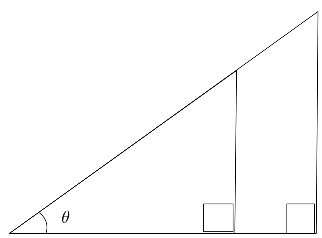
If we have two similar triangles, $\triangle\text{ABC} \sim \triangle\text{DEF}$, then we can think of putting the bigger one inside of the smaller one and we have the following relations:



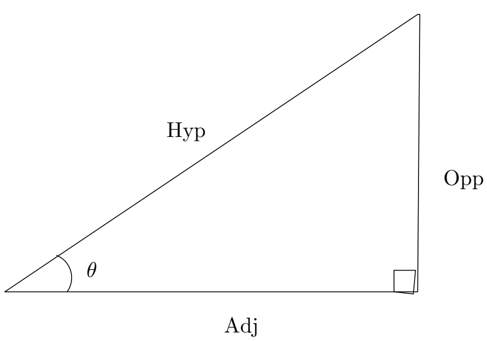
$$\frac{|\text{AE}|}{|\text{AB}|} = \frac{|\text{AF}|}{|\text{AC}|} = \frac{|\text{EF}|}{|\text{BC}|}$$

Furthermore, we'll have EF parallel to BC (this follows from the alternate interior angle theorem).

It seems that all that really matters is the measure of the angle at A/D, so we might as well restrict our attention to similar right triangles, which will be determined completely by one of the non-right angles.



We can find invariants related to similarity by looking at ratios corresponding to the angle θ and we can give these ratios names.



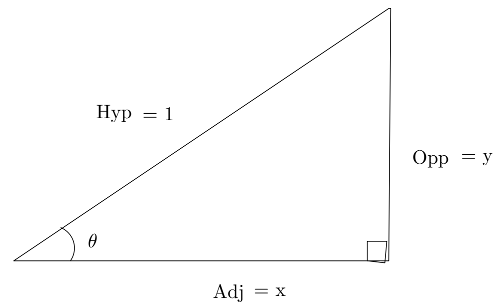
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

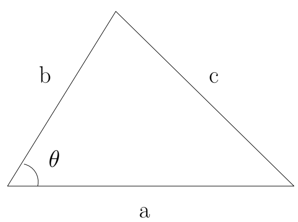
$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

Since these are ratios, we may as well choose the triangle with hypotenuse equal to 1. This will give $x = \cos\theta$ and $y = \sin\theta$.
Then, the well-known Pythagorean Theorem, tells us that $x^2 + y^2 = 1$ and so

$$\cos^2\theta + \sin^2\theta = 1.$$

Many other standard trig identities are just the Pythagorean Theorem in disguise.



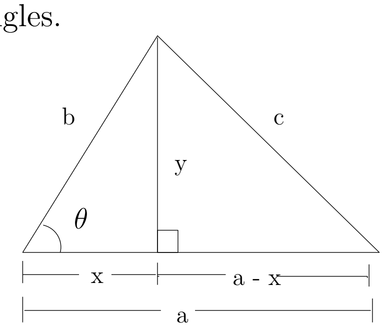


We can also derive a generalization of the Pythagorean Theorem called the **Law of Cosines**.
It says that in any triangle with side lengths a, b and c, we have

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where θ is the angle opposite the side of length c.

How do we prove this? We drop an altitude down and then use the Pythagorean Theorem on the two resulting right triangles.



For the triangle on the left, we get $x^2 + y^2 = b^2$ and for the triangle on the right we'll have $(a - x)^2 + y^2 = c^2$.

Solving the equation for the triangle on the left for y^2 will give us $y^2 = b^2 - x^2$. Now substitute this into the equation for the triangle on the right, yielding

$$(a - x)^2 + b^2 - x^2 = c^2.$$

Expanding the equation gives $a^2 + 2ax + x^2 + b^2 - x^2 = c^2$ and canceling the x^2 terms gives us

$$a^2 + 2ax + b^2 = c^2.$$

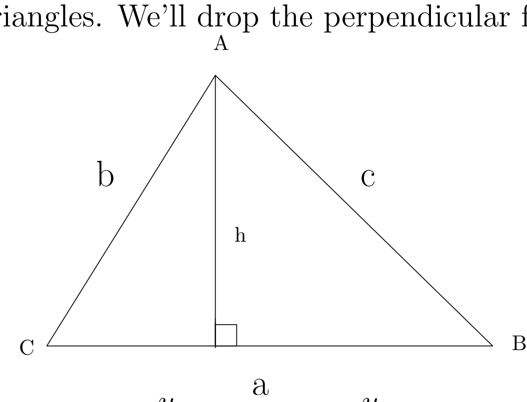
Now notice that $\cos\theta = x/b$; solving for x gives $b\cos\theta = x$. Making the substitution for x in the above equation gives us the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Also we can obtain the **Law of Sines**. This says that in any triangle, if the angle of measure A is opposite the side of length a, the angle of measure B is opposite side b, and the angle of measure C is opposite side c, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

How do we prove it? Again, drop an altitude down from an angle and convert the problem into one involving right triangles. We'll drop the perpendicular from angle A.



Notice that we now have $\sin B = \dfrac{y}{c}$ and $\sin C = \dfrac{y}{b}$ so, solving both equations for y gives $y = b \sin C = c \sin B$. Since $b \sin C = c \sin B$, we can divide both sides by b and c, to get

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

If we use the same process, dropping an altitude from angle C, we will get

$$\frac{\sin B}{b} = \frac{\sin A}{a}.$$

Combining these two gives the result.