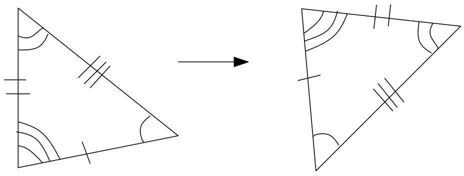
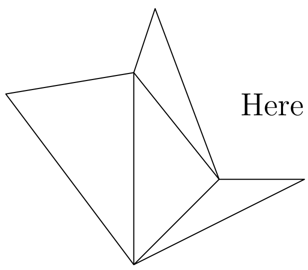
Congruence and Isometry

Congruence of triangles mean corresponding parts are equal. In laymen's terms this means we can cut one of the triangles out of the plan, move it around, and then lay it over the other triangle.



Why do we focus on triangles in secondary school (and earlier)?

- Right triangles are special and we use them to develop trigonometry.
- Triangles are easy to work with (sum of angles is 180°, sum of two sides is greater than the third side)
- We can make any other polygon out of trianges. In otherwords, triangles are the building blocks of polygons.



Here is a polygon that has been constructed out of triangles.

We talk a lot about when two triangles are the same. We have the traditional list of when triangles are the same:

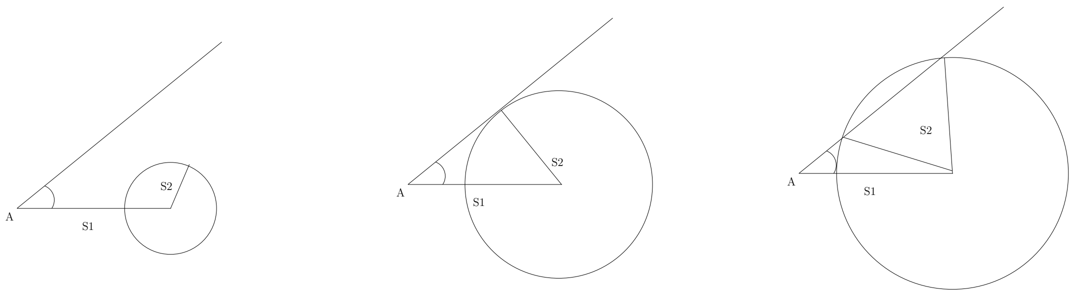
- SSS (side, side, side: the corresponding sides have equal measure)
- SAS (side, angle, side: the corresponding sides and included angles have equal measure)
- ASA (angle, side, angle: the corresponding angles and included side have equal measure)
- AAS (angle, angle, side: the corresponding angles and a non-included side have equal measure – this is basically the same as ASA)

But what about SSA (side, side, angle:the corresponding sides and a non-included angle have equal measure) or AAA (angle, angle, angle: the corresponding angles have equal measure)?

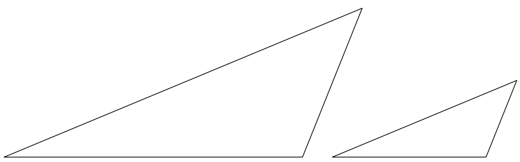
Given two side lengths and a non-included angle (SSA) there are three possibilities:

1. We can't build any triangle.
2. We can build exactly one triangle.
3. We can build two possible triangles.

Below are the pictures showing the three possibilities:



From AAA (corresponding angle measures are equal) we get similarity which means that corresponding side lengths, while not necessarily equal, are in ratio.



Notice that a congruence is the same thing as looking at figures modulo some relation. In our case figure$_1$ $\sim$ figure$_2$ if there is a motion from one to the other. What do we mean by motions? We mean distance preserving functions. That is, if x and y are two points on a figure and f is a function taking one figure to another then $d(x, y) = d(f(x), f(y))$.

Our motions are:

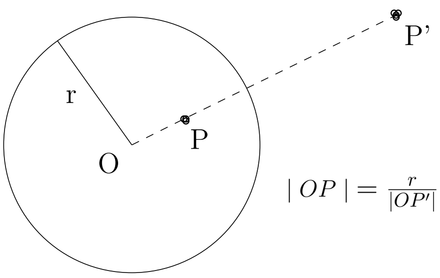
1. A rotation of θ degrees: $(x, y) \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

2. A relflection over a line: Fine the line perpendicular to given line which goes through your point then travel the same distance from the perpendicular line to the point in the opposite direction.

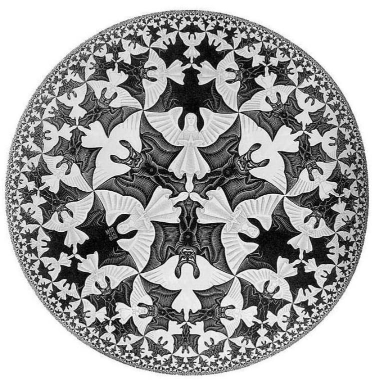
3. A translation: $(x, y) \longrightarrow (x + c, y + d)$ for some $c, d \in \mathbb{R}$

We can also consider other motions and call them isometries.

Here is an example:
We could fix a circle γ and, taking points other than the origin, take the inversion through the circle.



This changes our usual notion of congruence and what a line is. We get hyperbolic geometry from this.
M.C. Escher had a varied notion of congruence, often making art using the Poincare disk. In the picture below, each of the demons are congruence and each of the angles are congruent.



Considering some congruences gives us nice group structures. The following are some examples:
What if we were in $\mathbb{R}^2$ and we wanted to onsider the figures that are invarient by a $90°$ rotation. In otherwords, $(x, y) \longrightarrow (-y, x)$. Let's call this rotation f. Notice that $f(x, y) = (-y, x)$ but if we apply f again then, $f(-y, x) = (-x, -y)$, applying again, $f(-x, -y) = (y, -x)$ and applying a fourth time $f(y, -x) = (x, y)$. So by applying this rotation 4 times we get back to where we started. Thus if we let a be any point in $\mathbb{R}^2$ then we get the structure, $\{a, a^2, a^3, id\} \cong \mathbb{Z}_4$. So any object that is invarient by a $90°$ rotation must have 4-fold symmetry.

Something else we can consider is taking an object from the plane, moving it around, and putting it back where it started. Let's consider this with the equilateral triangle. How many ways can we take the triangle out and put it back. It stands to reason that there are 6, 3 that come from rotations and 3 that come from reflections. How do we know that those are all of the symmetries? We can pick a vertex. Then, face up, we have three choices of where to put the vertex back and face down we have another three choices. What's nice is that all 6 of these motions are generated by 1 rotation of $120°$ and 1 reflection. This is where algebra and geometry intersect. We can consider this with any regular polygon.

Next is the notion of wallpaper symmetry. There are only 17 kinds of symmetries that wall paper can have. See here for more information. Notice that there are only 4 main rotational symmetries in the wallpaper groups. They come from looking at pieces of wallpaper shaped like triangles, squares, hexagons, and octagons and gluing them together. Then our rotational symmetry comes from rotating each smaller piece of the paper. Why are these the only 4 shapes that work? We will discuss that next time.