MAE 301/501 notes

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Once again, we are reminded of this process of "building up to the reals"

$$\mathbb{N} \rightsquigarrow \mathbb{Z} \rightsquigarrow \mathbb{Q} \rightsquigarrow \mathbb{R}$$

But we also know there to be an analog between $\mathbb{Z}$ and polynomials, $K[x]$ for some field K, between $\mathbb{Q}$ and rational functions, $\dfrac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{n} b_i x^i}$, but what about the analogue of $\mathbb{R}$?

Maybe some possibilities are:

$$\sum_{i=0}^{\infty} a_i x^i \qquad \text{(formal power series)}$$

$$\sum_{i=-\infty}^{\infty} a_i x^i \qquad \text{(Laurent series)}$$

Note there are issues here if the series has a finite radius of convergence.

Just as we really only "see" rational numbers, but we know irrationals "must" be there (the diagonal of a unit square, for example, **must** have a length). It also stands to reason that this sort of thinking can be applied even further.

One such example is the Hyperreals (denoted by $^*\mathbb{R}$). The Hyperreals is the Reals augmented by the infinitesimals. Numbers in the Hyperreal set have a "standard" Real part and an infinitesimal part (comparable to how Complex numbers have a Real and Complex part). When Newton and Leibniz both developed calculus they did so using infinitesimals (Newton called them "fluxions") rather than limits. Unfortunately, there was a lot of trouble with these mystical quantities which were sometimes nonzero and sometimes treated as zero. Because improper use led to contradictions (although no one could say what use was proper), eventually the foundations of calculus were replaced with limits.

Infinitesimals were mostly discarded from mathematics, aside from a few failed proofs of their existence, until Abraham Robinson published *Non-Standard Analysis* in 1966, where he formalized the Hyperreals, and proved that these were a consistent extension of the real numbers. That is, Hyperreal numbers "exist".

Infinitesimals are numbers that are smaller than all real numbers. A non-zero infinitesimal ε satisfies: $0 < |\varepsilon| < x$ for every real number x. (Non-zero because 0 is considered tobe an infinitesimal, as well as an ordinary Real).

Some field properties of $^*\mathbb{R}$:
ε and δ are infinitesimal. $b \in^* \mathbb{R}$ is not infinitesimal.

- 0 is the only real number that is also infinitesimal
- All real numbers are finite
- $-\varepsilon$ is infinitesimal
- $\frac{b}{\varepsilon}, \varepsilon \neq 0$ is infinite
- $\dfrac{1}{\infty}$ is infinitesimal
- $\varepsilon + \delta$ is infinitesimal
- $b + \varepsilon$ is not infinitesimal
- $\varepsilon \cdot \delta, b \cdot \varepsilon$ are infinitesimal
- $\frac{\varepsilon}{b}, \dfrac{\varepsilon}{\infty}, \dfrac{b}{\infty}$ are infinitesimal

There is also the Standard Part which has several notations, depending on which book you read. The standard part "extracts" the Real value from a Hyperreal number. I will use the notation $st()$ for the standard part.

$$^*x \in^* \mathbb{R} \qquad st(^*x) = x \in \mathbb{R}$$

Here is an example of a derivative using hyperreals and infinitesimals instead of limits.

In this example dx and dy are infinitesimal (but non-zero) changes in x and y.

$$\begin{aligned}
f(x) &= x^2 + 5x + 10 \\
y &= x^2 + 5x + 10 \\
y + dy &= (x + dx)^2 + 5(x + dx) + 10 \\
dy &= (x + dx)^2 + 5(x + dx) + 10 - y \\
dy &= (x + dx)^2 + 5(x + dx) + 10 - (x^2 + 5x + 10) \\
dy &= x^2 + 2x(dx) + (dx)^2 + 5x + 5(dx) + 10 - x^2 - 5x - 10 \\
dy &= 2x(dx) + (dx)^2 + 5(dx) \\
\frac{dy}{dx} &= \frac{2x(dx) + (dx)^2 + 5(dx)}{dx} \\
\frac{dy}{dx} &= 2x + dx + 5 \\
st\left(\frac{dy}{dx}\right) &= st(2x + dx + 5) \\
f'(x) &= 2x + 5
\end{aligned}$$

As you can see, by doing this we can find derivatives with algebra alone, no limits.

The point of this discussion was to elaborate on the idea that while we do our work with $\mathbb{R}$ or $\mathbb{C}$, we can still extend the notion of these kinds of numbers. Of course, it remains to be seen whether this extension process leads to a useful extension of the idea of a number. Complex numbers were in a similar state about 200 years ago: their existence was shown, but they weren't "honest numbers" until the were shown to be useful.

It isn't at all clear, but perhaps there might be an analogue between the extension of power series to more general functions and the extension of the reals to the hyperreals. This is, of course, purely speculative.

(Euclidean) Geometry:
Euclid had 5 postulates and 5 "common notions".

1. 2 points create a unique straight line
2. a line segment can be extended in a straight line
3. a center point and a distance (radius) creates a circle
4. all right angles are equal
5. given 2 lines and a transversal, if the 2 interior angles on the same side of the transversal are less than right angles, then the 2 lines will meet at some point

Geometry can be done with an axiomatic development. However, high school geometry often does not distinguish between axioms, theorems, and assumptions. This leads to student problems later when asked to prove something they've always been told is true. Students think that the word "Geometry" means "two-column proofs". We, as future teachers, must strive to not let our students think that. Also, it is important to emphasize the difference between drawing conclusion from fact rather than experimentation.