MAE 501 March 3, 2009

Homework problem:

Find all digits to the repeating decimal 1/19 = 0.052631578947368421 using a calculator.

Katie's way…

On calculator, we find the multiples of 1/19:

1/19 ≈ **0.05263157 8**9
2/19 ≈ 0.10526315789
3/19 ≈ 0.1578**94736**8
4/19 ≈ 0.2105263158
5/19 ≈ 0.2631578947
6/19 ≈ 0.3157894737
7/19 ≈ 0.36**8421**0526
etc.

Above, the numbers in bold are the subsequent numbers in the decimal of 1/19. These 18 numbers will continue to be in this order in any multiple of 1/19. This is because 1/19 is known as a cyclic number, defined as an (n-1)-digit integer that, when multiplied by 1, 2, 3,…, n-1, produces the same digits in a starting in different order. They are generated by the full repetend primes, or primes for which 1/p has a maximal period decimal expansion of p-1 digits , i.e. 7, 17, 19, 23, 29, 47, 59, 61, 97,…

Deana's Way…

This way is similar to Katie's way. Let's show with 1/7, which is equal to 0.142857, repeating. If we had a calculator that only could find 4 digits, 0.1428, how would we use this same calculator to find the remaining digits, before it repeats?

First, we divide and get as many digits as we can. Since we have a 4-place calculator, we get:

1/7≈ 0.1428

Now multiply our answer by 7, and subtract from 1:

1 - (0.1428 * 7) = 1 – 0.9996 = 0.0004

Now we repeat the process with our 4-place calculator, to determine what 4/7 is:

4/7≈ 0.5714

We can see that it is repeating, however, if we want to do one more round to be sure, we can.
This time we subtract from 4:

4-(0.5714 * 4) = 4 – 3.9998 = 0.0002

Thus, the next section will be whatever we get from 2/7:

2/7≈ 0.2857

Putting these together, we have

1/7≈ 0.1428 5714 2857

That is, 0.142857 repeating.

This is basically the same as long division - when we do it by hand, we compute one digit at a time, and with the calculator we are able to compute many digits (about 7-10) at one time. Just to clarify, I'll write out the calculation of 1/19 in the usual division tableau, listing the 10 digits at a time my calculator gives (staggering the lines so you can tell what's going on)

```
          0.          4736842105
          ___0526315789__________2631578947_____
     19  | 1.0000000000
            .9999999991
                        90000000000
                        89999999995
                                   50000000000
                                   49999999993
```

Of course, the last multiplication wasn't really necessary, since we see that it had begun to repeat. Notice the similarities between this and Katie's method above. In Katie's method, all of the divisions of n/19 are calculated, and then the appropriate digits are picked out. In the "long division" method (Deanna's method), we look at the digits of just those that we need.

Question: Which method would you use to find the digits of 1/199, which has an expansion of

1/199=0.00502512562814070351758793969849246231155778894472361809045226130653266331
6582914572864321608040 2 (repeating)

The Long Division Algorithm is very useful, but in the last several years is downplayed in schools due to an increased reliance on calculators. What we have shown with this problem is that even with the calculator, we are only able to get to a certain point. The knowledge about long division has enabled us to compute the digits beyond one calculation on the calculator. Furthermore, this algorithm is very useful in other contexts, which we see again when we look at division of polynomials.

Back to Complex Numbers

To get the Complex Numbers, **C**, we take the Real Numbers **R** and add ***i*** ($\sqrt{-1}$) **.**

C = **R**[*i*] = {a+b***i*** | a,b ϵ **R**}

This is a field extension. Another example of a field extension would be to take the rationals, **Q,** and adjoin $\sqrt[3]{2}$, giving us **Q**[$\sqrt[3]{2}$], which are all possible combinations of rationals and $\sqrt[3]{2}$, as well as powers of $\sqrt[3]{2}$. This means

$$\mathbf{Q}[\sqrt[3]{2}] = \{ a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbf{Q}\}$$

We only need to add rational multiples of $\sqrt[3]{2}$ and $\sqrt[3]{4}$ (and no others), because $(\sqrt[3]{2})^3$ is again a rational number, and $1/\sqrt[3]{2} = \sqrt[3]{4}/2$, also a rational multiple of $\sqrt[3]{4}$. On the other hand, if we want to consider **Q**[π], we need to add in all the rational multiples of all the powers of **π.**

When high school students think about building the Real Numbers, **R,** they typically think that you can just throw in the rationals , square roots, and maybe multiples of π and e, and then you get the Real

Numbers. However, there are many sub-groups added to make **R**, e.g. **Q**[√2], **Q**[√5], as well as uncountably many other transcendental numbers.

However, to build the Complex Numbers, it is a much easier concept for our students - you simply add the number *I* = √-1 to the Real Numbers, as well as all of its multiples. We don't need to add more, since i^2=-1 and 1/*i* = -i. This is easy for the students to understand because the notion of the Complex Numbers is new and easier than the notion of the Real Numbers, which is more complicated, older and harder to grasp.

A Complex Number is written in the form: a + b*i*, where a and b are Real Numbers, and *i* is the imaginary number (√-1). Complex Numbers can be added, subtracted, multiplied and divided as Real Numbers can be, using the associative, commutative, and distributive laws of algebra. This makes the Complex Numbers a field.

- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$
- Multiplication: $(a + bi) \bullet (c + di) = ac + bci + adi + bdi^2 = (ac-bd) + (bc + ad)i$
- Division: $\frac{(a + bi)}{(c + di)} = \frac{(ac + bd)}{(c^2 + d^2)} + \frac{(bc - ad)}{(c^2 + d^2)} i$

Because the Complex Numbers are a field they also have the following:

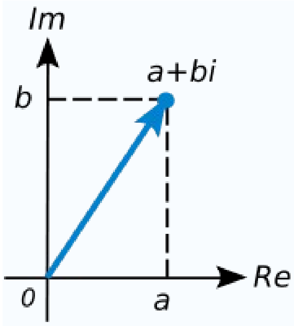
- Additive identity (zero): $0 + 0i = 0$
- Multiplicative identity (one) : $1 + 0i = 1$
- Additive Inverse (-a + b*i*): $(a + bi) + (-a - bi)$
- Multiplicative Inverse $\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i \right)$: $(a + bi) \bullet \left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i \right) = 1$

It is important to note, however, that the complex numbers cannot be made into an *ordered* field.

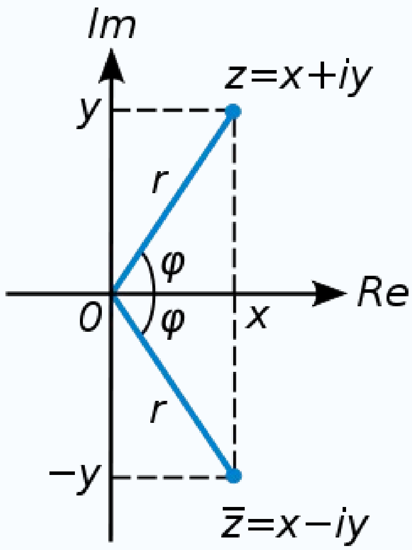
Because of the natural isomorphism between the complex numbers and pairs of real numbers given by

$$a + bi \leftrightarrow (a,b)$$

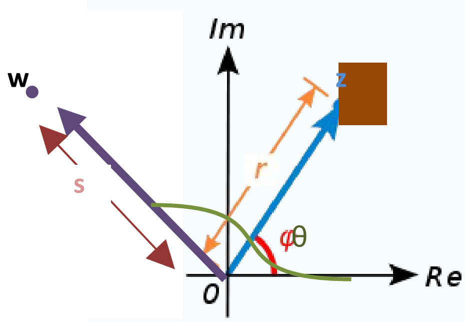
we can view a complex number as a point in the plane (or as a vector). Below we have a Complex Number shown on this graph as a vector, where the horizontal axis is the Real Numbers and the vertical axis is the Imaginary Numbers.



A vector z can be represented with its complex conjugate in the complex plane.



Addition of Complex Numbers corresponds to vector addition, and multiplication of Complex Numbers is multiplying the lengths of the corresponding vectors and adding the angles that the vectors make. This can be shown using polar coordinates.



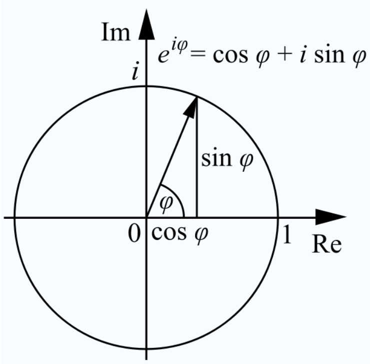
Just to confirm this, let's take two vectors z and w…

z = a + b*i*, which is the vector with length r, and angle ϕ.
This can be written in polar coordinates as: z = r (cos ϕ + *i* sin ϕ) because a = r (cos ϕ) and b = r (sin ϕ)

Similarly, w can be written as w = s (cos θ + *i* sin θ), where the vector length is s, and the angle produced is θ.

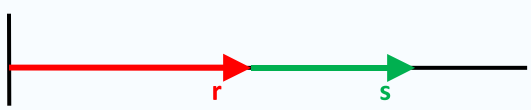
Multiplying the vectors, we find:
z●w = r (cos ϕ + *i* sin ϕ) ● s (cos θ + *i* sin θ)
= rs (cos ϕ cos θ + sin ϕ sin θ + *i* (sin ϕ cos θ + sin θ cos ϕ)
= rs (cos(ϕ + θ) + *i* sin(ϕ + θ))

Using Euler's formula, we can write z = r (cos ϕ + *i* sin ϕ), as z = $re^{i\varphi}$. This formula, developed by Leonard Euler, shows a relationship between the exponential function e and the trigonometric functions, as shown geometrically below.



In this form, it is easier to see that zw = $(re^{i\varphi})(se^{i\theta}) = (rs)\, e^{i(\phi + \theta)}$, that is, the lengths of the vectors are mulitplied and the angles are added.

It is easy to see that addition of Complex Numbers corresponds to our definitions for the Real Numbers, using the geometric interpretation with vectors strictly on the x-axis.



We can look at multiplication of real numbers the same way.
Product of two positives is positive: r●s = r●s (cos 0 + *i* sin 0) = rs
product of a positive and a negative is negative: r●(-s) = (r cos 0) ● (s sin π) = -rs
product of two negatives is negative: (-r)●(-s) = (r cos π) (s sin π) = rs (cos 2 π) = rs

Weisstein, Eric W. "Cyclic Number." From MathWorld –A Wolfram Web Resource, http://mathworld.wolfram/CyclicNumber. html

Figures courtesy of Wikipedia, The Free Encyclopedia – ***Complex Number*** and ***Euler Formula***