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MAE301 Class Notes
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Counting with repeating decimals:

Recall that the decimal for p/q terminates after t places if and only if $q = 2^r 5^s$, where $t = max(r, s)$. In the last class we showed that an eventually repeating decimal is a rational number. We proved that a rational number can be a repeating decimal, and that eventually repeating decimals are rational by noticing that only finitely many $(q-1)$ possible remainders before having to reuse any.

So why *are* decimals useful?
Some advantages of decimals:

- Decimals make it easier to see the quantitative value of a number. That is, they make relative comparisons easy.
  For example: we know that $.25 > .2$ just by looking at it, but it isn't so easy to make this distinction when the decimals are in their equivalent fraction form ($1/4 > 1/5$). A less obvious example would be comparing 11/17 and 24/37;
- Decimals also make it easier to approximate.
  For example, we have a complicated looking fraction 221/1973 which is approximately 22/197 which is approximately .112. In this case, it is easy to see how less complicated the decimal is to understand.
- Decimals are also better to use for calculations. You can add and subtract decimals in the same fashion as you would for the integers, but when you add or subtract a fraction, you first have to find a common denominator before you can do anything else.

Some disadvantages of decimals:

- Decimals are not always as succinct, or concise.
  For example, consider the fraction 1/19. This equals .052163158947368421052163...
  This long repeating decimal is much more consise in fraction form.
- There can also be some confusion when representing a number in decimal form.
  For example: The numbers $123,456$ and 123.456 can be easily confused. This is especially problematic when Europeans and Americans are both involved, since Europeans use "," for the decimal point.
- Significant figures, sometimes called "sig-figs", may also cause some confusion. Sig-figs are typically an issue when dealing with measurements in science. The issue arises because students often don't understand the purpose.

For example, when employing significant figures two represent a measurement of 2 units to within $\pm.05$, this is written as 2.0. If we were to add a more precisely measured quantity, say 1.234, we would claim

$$2.0 + 1.234 \approx 3.2,$$

which is correct to two significant figures. However, students often get confused about when to add more figures and when to drop them.

When we have a long rational decimal, what exactly does it mean? Take, for example, the number 2.94723. By this we mean $2 + \frac{9}{10} + \frac{4}{100} + \frac{7}{1000} + \frac{2}{10000} + \frac{3}{100000}$.

However, we don't necessarily have to represent numbers using powers of 10. In some cases, it might be more convenient to use powers of two, or three, or some other convenient base.

For example, we could represent the number $3/4$ in base 10 as 0.75, by which we mean

$$\frac{3}{4} = \frac{7}{10} + \frac{5}{100}.$$

However, we could also use powers of two:

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}.$$

In this case, we could write this as a "base-2 decimal" or "bicimal", that is,

$$\frac{3}{4} = 0.11_{[2]}$$

We can generalize this to any number. More precisely, every real number x between 0 and 1 can be written uniquely as

$$x = \sum_{i=1}^{\infty} \frac{a_i}{2^i},$$

where each of the numbers a_i are either 0 or 1 (and we make the convention that we don't end in an infinite string of 1s). We can write this in base 2 as the string

$$0.a_1a_2a_3a_4\ldots_{[2]}$$

(if all of the a_i beyond some point are 0, we usually just stop writing them and terminate the expansion.)

This is in exact analogy to our usual decimal representation, where we write

$$x = \sum_{i=1}^{\infty} \frac{b_i}{10^i} \quad = 0.b_1b_2b_3b_4\ldots$$

(with the convention that if all the b_i are 0 after some point, we terminate the decimal, and we don't write a decimal ending in all 9s.

These base-2 decimals are what is used internally in computers, for example.

Note that in base 2, the decimal will terminate exactly when the fraction represented is a dyadic rational, that is, can be written in the form $p/2^k$. The consequence of this is that many of our terminating (base 10) decimals cannot be represented exactly in a computer. For example, in base 2, 1/10 does not have a terminating representation. Rather, it is has a repeating expansion, since

$$\frac{1}{10} = 2^{-4} + 2^{-5} + 2^{-8} + 2^{-9} + 2^{-12} + 2^{-13} + \ldots = 0.0001100110\overline{0011}_{[2]}$$

How can we determine the base-2 representation of a number? Let's go through it for 1/10 a little more carefully.

First, $1/10 < 1/2$, so the first digit is 0 (we have no halves).

Next, $1/10 < 1/4$, so the next digit is also 0 (no quarters, either).

Also, $1/10 < 1/8$, so the next is 0 as well.

But, $1/10 > 1/16$, so the next digit is 1.

Now it gets a little more complex. We now compare $1/10 - 1/16$ to 1/32 (it is still bigger), so the next digit is 1 as well, and then we look at how $1/10 - 1/16 - 1/32$ compares to 1/64, and my head is starting to hurt.

Let's try to be a little smarter, and start over.

$1/10 < 1/2$, so the first digit is 0. Now, rather than halving 1/2, let's double the 1/10, and compare that to 1/2 (doubling corresponds to shifting right one place). When the result of doubling is bigger than 1/2, we get a 1 in the expansion. Then we subtract 1/2 from the number we are trying to represent, and continue.

Let's try:

$$\begin{array}{cccccccccccccc} \frac{1}{10}, & \frac{2}{10}, & \frac{4}{10}, & \frac{8}{10} \rightarrow & \frac{3}{10}, & \frac{6}{10} \rightarrow & \frac{1}{10}, & \frac{2}{10}, & \frac{4}{10}, & \frac{8}{10} \rightarrow & \frac{3}{10}, & \frac{6}{10} \rightarrow & \frac{1}{10}, & \ldots \\ .0 & 0 & 0 & & 1 & & 1 & 0 & 0 & & 1 & & 1 & \ldots \end{array}$$

Here it is clear the pattern will repeat.

We can adapt this algorithm to represent a number in other bases. For example, to get 1/4 in base three (here we triple, and compare to 1/3 or 2/3):

$$\begin{array}{ccccc} \frac{1}{4}, & \frac{3}{4} - \frac{2}{3} \rightarrow \frac{1}{12}, & \frac{3}{12} = \frac{1}{4}, & \frac{3}{4} - \frac{2}{3} \rightarrow \frac{1}{12}, & \ldots \\ .0 & 2 & 0 & 2 & \ldots \end{array}$$

So $1/4 = 0.02\overline{02}_{[3]}$, which we already knew since

$$\sum \frac{2}{9^n} = \frac{1}{4}.$$