MAE 301
2/10/09 Class Notes

Last Class

$\mathbb{R}$ can de defined in the following ways

- Equivalence classes of Cauchy sequence
- Points on an infinitely long number line (depending how one thinks about points on the line)
- Dedekind cuts
- Infinite decimal representation

Let us think about how we would go from $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$?

If we want to go from $\mathbb{N} \to \mathbb{Z}$ what we would do is take all the natural numbers and add few new things to them, in this case it would be their negatives and get the integers. Similarly, if we want to go from $\mathbb{Z} \to \mathbb{Q}$ we would take the integers and introduce quotient of them to obtain the rational numbers. We call this extending the set. However, note that there is a bit of an issue in that we are making up these things out of whole cloth, so to make sense of these "new" numbers (negatives, irrationals, etc.), we must define them completely in terms of the already known numbers. Thus, to define "-2" as "what you get when you take 2 away from 0" is no definition at all.

Dedekind cuts

A Dedekind cut is the partition of the rationals (a totally ordered set) into two non-empty parts. It is named after Richard Dedekind.
Let us pick a point on the number line and cut the line at that point. Now we will examine how the cut relates to $\mathbb{Q}$.

Left Side   Cut   Right Side

The made cut divides $\mathbb{Q}$ into two sets, "right" and "left" side
or
"big" and "small"
such that $\forall x \in$ "left" side and $\forall y \in$ "right" x>y
We have several possibilities of how the cut falls:

1) Left side has a largest element, right side does not have a smallest elements.
2) Left side has no largest element, right side has a smallest element.
3) Left side has no largest element, right side has no smallest element.

If you cut on rational number, you get a rational number. If you cut between rational numbers, you will get something new. Note that cases 1 and 2 above correspond to when we cut at a rational number. Case 3 corresponds to irrational numbers.

The difficulty is to check that each of these “new” kinds of numbers satisfy all of the field axioms. We need to define how to do arithmetic with these sets of rationals, and so on. We won't, however, do that.

New material

What is an equation?
An equation is simply a mathematical statement written in symbols, stating that two things are exactly the same. One can think of an equation as an English sentence, which is written with some symbols. Another way to interpret an equation is in a form of a story. In order to fully understand the story presented to us, we need to be aware of the previous occurrences. It works the same way with the equations. We need to understand basic properties and meaning of things, before we can approach the equation.

When we look at the equation we usually do not think to express it in English, we just simply proceed in solving it. What if someone looked at it and simply saw some symbols, which had no meaning to them? We need to be able to explain it to them in a form of a sentence because after all the equation is an existence statement.

Let us consider the following equation

$$3x+5=11$$

Translating it into words, we obtain:
Consider all real numbers x, so that three times that number plus five equals eleven.
When we say that the equation has a solution, we are asserting that the set of numbers described by the equation is nonempty.

Now let us look at the steps in solving the equation.

$$\begin{aligned} 3x+5&=11 \\ -5\ \ &-5 && \text{(Step I)} \\ 3x&=6 \\ 1/3(3x)&=1/3(3x) && \text{(Step II)} \\ x&=2 \end{aligned}$$

(Step I) is allowed because subtracting/adding the same thing from both sides of the equation preserves the equality.
(Step II) is allowed because multiplying or dividing by nonzero number on both sides on the equation preserves the equality as well.

Let us look at another equation

$$\begin{aligned} 3x^2+3&=6 \\ -6\ \ &-6 && \text{(Step I)} \\ 3x^2-3&=0 \\ x^2-1&=0 && \text{(StepII)} \\ (x-1)(x+1)&=0 \\ x-1=0 \qquad x+1&=0 && \text{(StepIII)} \end{aligned}$$

$x=1$ or $x=-1$ (Step IV)

From this example (Step I, II, and IV) were explained in the previous example, the only one which might not be clear is (Step III). This step is allowed because ab=0 implies that one of the numbers a or b has to be zero. This is a crucial property of the reals: there are no zero divisors.

Now let us consider another example and see that one needs to be very careful when teaching student how to solve equations. Each step has to be justified clearly by different properties. The following example shows the danger when those justifications were not made.

$$3x^2-3=6$$
$$3(x^2-1)=6$$
$$(x^2-1)=2$$
$$(x-1)(x+1)=2$$
$$x-1=2 \text{ or } x+1=2$$
$$x=3 \text{ or } x=1$$

Clearly this is BAD!!!

Reason why we were able to do (Step III) in the previous example is because the equation was set to equal zero. In this example the equation is set to equal two, and this is not worthwhile, since if ab=2, we can't conclude that either a=2 or b=2. However, this is a very common mistake among students, and needs to be watched for carefully. Merely stating that this “is not allowed” does not instill a feeling for why mathematics works.

Equations take on different meaning depending on what it is that we are trying to solve for. If we look at the equation, $3x^2+5=1$ and just say “solve for x” we usually just do algebraic manipulation and find out the value(s) of x. However, we know what kind of solution we are interested in. When teaching we should get in the habit of explain to students what kind of x it is that we are looking for. Without that information, we can come up with different answers depending on if we look at x in terms of $\mathbb{R}, \mathbb{N}, \mathbb{Q}$, etc.

Let us look now on what it means to take the square root of a number, and how we have to be alert when dealing with the square roots in equations.
Here is an example when all the steps are correct and clearly the answer is correct.

$$\sqrt{x}+1=2$$
$$\sqrt{x}=1$$
$$(\sqrt{x})^2=(1)^2$$
$$x=1$$

Now here is another example where all the same steps are repeated.

$$\sqrt{x}+3=2$$
$$\sqrt{x}=-1$$
$$(\sqrt{x})^2=(-1)^2$$
$$x=1$$

This is clearly not the answer because there are no real roots to that make this equation work.

The issue here is that we have supressed the logic when solving. That is, what we are really doing is more like the following:

1. (statement) Find all real values of x so that $\sqrt{x}+1=2$
2. (1) holds if and only if $\sqrt{x}=1$, that is, these statements are equivalent.
3. If (2) holds, then we must also have $(\sqrt{x})^2=(1)^2$
4. Consequently, if x is a solution to the original equation, then it must be equal to 1.

Note that this means that 1 is the only possibile solution, not that it is actually a solution. We need to check it to be sure.

Looking at the second example, we see that it is equivalent to the statement,

If x is a real number for which $\sqrt{x}+3=2$, then we must have x=1.

However, there is no guarantee that there is actually such an x. By analogy, the statement “If my grandmother had wheels, she would be a truck” is true, since my grandmother does not have wheels.

We need to carefully keep in mind which steps in solving an equation are equivalences (that is, reversible), and which ones are hypothetical implications (not reversible). Whenever we use the latter type, we need to be careful and confirm that such solutions are not “spurious”.+