Chan, Yui Suen
Class Notes for 02/05/2009

Field of Rationals

In mathematics, a field is a set F. together with two operations, called addition **+**, and multiplication, $\cdot$, such that the following axioms hold [2]:

1. (*closure*) Addition and multiplication are binary operations such that for all a and b in F, $a + b$ and $a \cdot b$ are also in F.
2. (*associativity*) The operations of addition and multiplication are associative: for all a, b, and c in F. we have $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
3. (commutivatity) The operations of addition and multiplication are commutative for all a and b in F, $a + b = b + a$ and $a \cdot b = b \cdot a$.
4. (distributive) Multiplication is distributive over addition: for all a, b, and c in F. we have $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
5. (identity) There exists an additive identity element, which is denoted by 0, such that for any element a, $a + 0 = a$. In addition, there exists a multiplicative identity element, denoted by 1, such that $a \cdot 1 = a$.
6. (inverse) For each element a, there exists an additive inverse element, denoted by is $-a$, such that $a + (-a) = 0$. In addition, for each element a **except** 0, there exists a multiplicative inverse element, denoted by a^{-1} or $\frac{1}{a}$, so that $a \cdot (a^{-1}) = 1$.
   In some cases (typically when the field is the reals or a related field), we call addition of an additive inverse *subtraction* (which we write omitting the *+,* as $a - b$ *instead of* $a + (-b)$*,* and multiplication by a multiplicative inverse is called *division*, written a/b.

One of the primary examples of a field is the set of rational numbers $\mathbf{Q}$. Recall that a rational number can be expressed as a fraction or a quotient of two integers [3]. Symbolically, the rational number can be expressed as $\mathbf{Q} = \{p/q : p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$, where **Z** is the set of integers [3]. We remark that here we have not been careful to ensure that the representation of a rational number p/q is unique. For example, 1/3 and 3/9 both represent the same rational number. If we want to ensure this (as is typically done), we say that a pair or rational numbers p/q and r/s are equivalent if $ps = rq$; we then prefer the representative with the smallest positive value of q. (This is called "writing the fraction in least terms", or "reducing" it).

As we saw last time, attempting to extend Q to a new set $\overline{\mathbf{Q}}$ which is both an additive and multiplicative group leads to additional issues. We can't just "define" division by zero so that it makes sense, without introducing a lot of other issues.

Discussion of homework problem

Question: What is wrong with trying to define addition of rationals in the naïve way, so that $\boldsymbol{p/q + r/s = (p + r)/(q + s)}$? What are the consequences of trying anything different from the usual definition $\boldsymbol{p/q + r/s = (ps + rq)/qs}$?

First, let's address the problem of naïve addition. If addition of fractions worked this way, the naïve definition gives $\frac{p}{q}+\frac{p}{q}=\frac{2p}{2q}$. But since $2pq=2qp$, we must have $\frac{2p}{2q}$ equivalent to $\frac{p}{q}$. That is, adding any rational to itself gives the same rational back. This also leads to trouble with the distributive law: $2\frac{p}{q}=(1+1)\frac{p}{q}=\frac{p}{q}+\frac{p}{q}=\frac{2p}{2q}=\frac{p}{q}$

(We remark that this kind of addition is actually useful in some contexts, and is called Farey Addition. However, it is more a kind of averaging than addition in the regular sense.)

We now turn to the more general question: why must addition be $\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}$? The answer has to do with both how we define multiplication and that we want the subset **Z** of $\boldsymbol{Q}$ to behave "the same". Let us first examine multiplication: the closure axiom tells us that the product of two rationals must be another, that is, we have $\frac{p}{q}\cdot\frac{r}{s}=\frac{a}{b}$, for some integers a and b. We have to show that $a=pr$, and $b=qs$. In addition, we want to have multiplication in **Z** correspond to multiplication in $\boldsymbol{Q}$. So, given an integer $n \in \boldsymbol{Z}$, we see that if this corresponds to a rational $\frac{p}{q}$, we must have $p=n$ and $q=1$. If $q=s=1$, then $\frac{\mathrm{p}}{1}\cdot\frac{\mathrm{r}}{1}=\mathrm{p}\cdot\mathrm{r}=\mathrm{pr}$. This means that if we want the same formula to work for rationals which are not equivalent to integers, we must have $a=pr$; continuing this argument shows that $b=qs$.

Use of this, together with a similar argument shows that we have no choice but to define addition of rationals as $\frac{p}{q}+\frac{r}{s}=\frac{ps+qr}{qs}$. Since it is a homework assignment, I'll omit the details here.

<u>How do we build real number from rational numbers?</u>

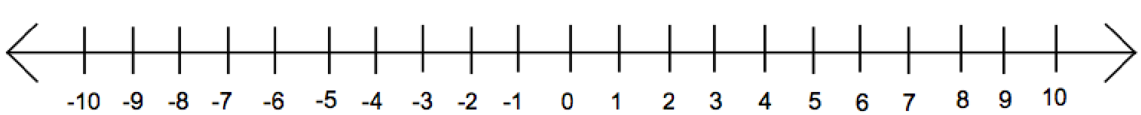
One of the ways to look at the establishment of the numbers is the elements of the set such that

$$\boldsymbol{N} \subset \boldsymbol{Z} \subset \boldsymbol{Q} \subset \boldsymbol{R}$$

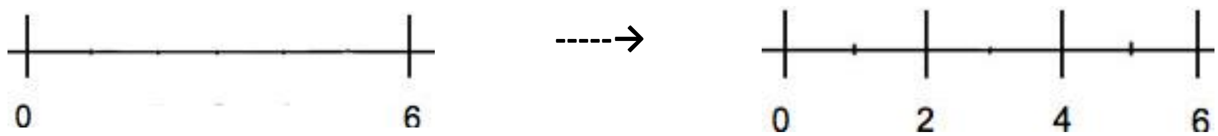
where

N: The set of natural numbers, (1, 2, 3, 4, 5,….)
Z: The set of integers, (….,-3, -2, -1, 0, 1, 2, 3,….)
Q: The set of the rational numbers (…., -8, -1/4, 0, 1, 2/3, 4.76, 5.09, 6,….)
R: The set of real number, (…., -8,$-\sqrt{2}$, -1/4, 0, 1, $\sqrt{3}$, 6,….)

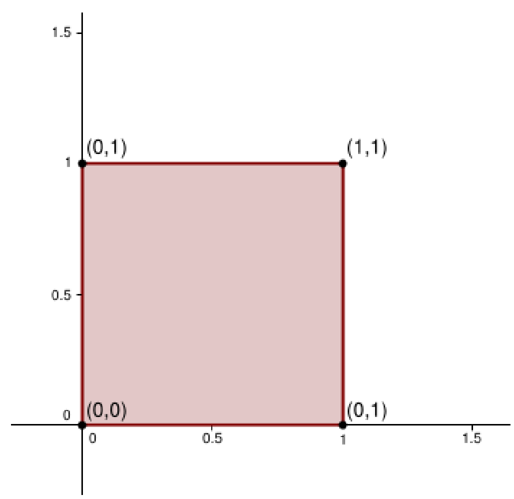
A standard metaphor for the reals is that each real number corresponds to a definite length. We arrange these lengths along a straight line (designated as the x-axis), and this axis is graduated so that each point is associated with a number. Take two arbitrary points on the line as the positions for 0 and 1 such that the distance between these two points is the unit length.



The 0-point is called the origin. The points to the right of the origin are the positive numbers, and the points to the left of the origin are associated with negative numbers (that is, the additive inverses of the positive numbers). In this way a number is attached to each point, the number being the distance from the point to the origin either with a plus sign if the point goes to the right or with minus sign if the point goes to the left. Rational numbers such as -1/4, 0, 1, 2/3, 4.76 and 9 are readily located by their relation to the unit length. For example, one of the way to think of 6/3 is



We extend this metaphor to obtain all rationals as ratios of lengths. But as has been known for thousands of years, there are more lengths than can not be expressed as ratios integers.



For example, the symbol $\sqrt{2}$ represents a number which would yield 2 when multiplied by itself. That is, $\sqrt{2} \cdot \sqrt{2} = 2$. In geometry, we consider a unit square (with side lengths 1 and 1); from the Pythagorean Theorem, we know that the length of the diagonal is $\sqrt{1+1}$. Therefore, we represent the length of the diagonal by $\sqrt{2}$ and associate the number $\sqrt{2}$ with that point on the line whose distance from the origin is equal to the length of the diagonal of our unit square [4].

This method gives us some of the real numbers, but there are many other reals which cannot be constructed in this way. In fact, we can only obtain a small subset of the reals in this way, called the *constructible numbers*.

One way to go from $\boldsymbol{Q}\ to\ \boldsymbol{R}$ is via Cauchy sequences. Recall that Cauchy sequence is a sequence whose elements become arbitrarily close to each other as the sequence progresses [1]. In other words, suppose ε is chosen as a positive real number. Beginning with a Cauchy sequence and eliminating terms one by one from the beginning, small ε is any pair chosen from the remaining terms will be within distance ε of each other after a finite number of steps [1]. Let's consider a sequence $\{S_n\}$, where $\{S_n\} = S_1, S_2, S_3, S_4, S_{5,} \dots \dots S_n$. $\{S_n\}$ is Cauchy if every $\varepsilon > 0$, there is an N so that for all $n, m \geq N$, we have $|S_n - S_m| < \varepsilon$. This is the Cauchy sequences of real numbers. We can define **R** as the collection of all Cauchy sequences of rational numbers, although as before, we have many Cauchy sequences which represent the same real number. We can say that two Cauchy sequences represent the same real number if “they have the same limit”, although this idea is a little circular (since the limit is what we are trying to define as a number.) Instead, we will say that two sequences $\{S_n\}$ and $\{T_n\}$ represent the same real if, for every $\varepsilon > 0$, there is an N so that for all $n \geq N$, we have $|S_n - T_n| < \varepsilon$.

One drawback of this approach is that, while rigorous, it isn’t a metaphor that high school students can relate to. Indeed, it is not one that most undergraduates relate well to.

Reference

1. "Cauchy sequence." *Wikipedia, The Free Encyclopedia*. 1 Feb 2009, 16:11 UTC. 10 Feb 2009 <http://en.wikipedia.org/w/index.php?title=Cauchy_sequence&oldid=267846300>.

2. "Field (mathematics)." *Wikipedia, The Free Encyclopedia*. 2 Feb 2009, 09:26 UTC. 10 Feb 2009 <http://en.wikipedia.org/w/index.php?title=Field_(mathematics)&oldid=268004339>.

3. "Rational number." *Wikipedia, The Free Encyclopedia*. 10 Feb 2009, 03:26 UTC. 10 Feb 2009 <http://en.wikipedia.org/w/index.php?title=Rational_number&oldid=269693166>.

4. "Unit square." *Wikipedia, The Free Encyclopedia*. 4 Feb 2009, 19:44 UTC. 10 Feb 2009 <http://en.wikipedia.org/w/index.php?title=Unit_square&oldid=268531417>.