January 29, 2009 Class Notes.

A brief recap from the first class (1/27):
What does a 10th grader think math is about?
Equations? Formulas? Numbers?

Several different sets of numbers were discussed:

- **N** – natural numbers. $\mathbf{N} = \{1, 2, 3, 4, \ldots\}$
- **W** – whole numbers. $\mathbf{W} = \{0, 1, 2, 3, \ldots\}$ or $\mathbf{N} \cup \{0\}$
  - The whole numbers is the set of natural numbers with one added element: the number 0.
- **Z** – integers. $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- **Q** – rational numbers. $\mathbf{Q} = \{m/n \mid n \neq 0, \quad m,n \in \mathbf{Z}\}$ (Bartle & Sherbert, 2)
- **R** – real numbers. $\mathbf{R} = \{x \mid x \text{ has an infinite decimal expansion}\}$
  - The real numbers can be thought of as the infinite number of points that can be found on a given line.
- **C** – complex numbers. $\mathbf{C} = \{x + iy, \quad x,y \in \mathbf{R}, \; i^2=-1\}$

Note that each set in the above list is contained in the next consecutive set in the list. So the natural numbers are contained in the whole numbers and the whole numbers are contained in the integers, and so on…
Also, the descriptions above are not quite complete. For example, are 3/5 and 6/10 different rational numbers? What about 0.999999... and 1.00000... ?

But what do these sets of numbers mean? How does the set of natural numbers compare to {World Series, car, 45, dog, Super Bowl, ...}? To be a “number” there is more required than just a set.

Let us start with the natural numbers.

The natural numbers can be considered as the numbers used for counting. For example, a set is countable if a natural number can be assigned to each element in the set. Thus the set can be “counted” using the natural numbers. But the idea of “number” includes a way to combine them, that is, addition.

We need to adjust our definition of natural number to include the basic operation of +. Note that this idea is built-in to our notion of what counting means: we can count a few things, then count some more things, or we can count the whole group as one. That is, we can add together the number of things in the set. We can always add one more thing and count the new group, and the order in which we count things doesn't matter-- we always get the same answer. Without the ability to add (or “count more”), a number isn't really a number.

That is, we have a set (the numbers 1, 2, 3, 4, ...) together with an operation (“counting more stuff”, i.e. +); these form a *semigroup*.

Let's define the term semigroup. A semigroup is a pair consisting of some set and some operation on that set, which follow certain rules. Denote the set by S and the operation by *. A semigroup is a pair (S, *) so that

- (closure) for every $x,y \in S$, we have $x*y \in S$.
- (associativity) for every $x,y,z \in S$, we have $(x*y)*z = x*(y*z)$

Note that a semigroup need not be commutative (ie, we need not require that $x+y = y+x$), although **N** is commutative (as are all the other sets of numbers we are discussing at the moment).

Note that the set of natural numbers is a countable, infinite set, unbounded in one direction. We can "build" the natural numbers by starting with 1 and repeatedly adding 1 to get the rest. This is the inductive property which defines **N**.

It is through the use of repeated addition on the set of natural numbers that students first learn about multiplication.
Example: 4 x 3 = 4 + 4 + 4 = 12 = 3 + 3 + 3 + 3 = 3 x 4

I want to point out here that the way mathematicians think about multiplication and the way beginning students (or most non-mathematicians) think of multiplication are somewhat different. Mathematicians usually think of addition and multiplication as being two different operations on our set that are related by distributive laws:

- $a*(b + c) = a*b + a*c$
- $(a + b)*c = a*c + b*c$

But students usually think of the distributive laws as a property of multiplication and addition that is "discovered to be true". This is a theme we will return to throughout the course. We'll come back to multiplication in a little while.

We can add the element 0 to **N** to obtain the "whole numbers" which are variously denoted by **W**, $\mathbf{N_0}$**,** or sometimes just **N**. (That is, some people include 0 as a natural number, and some don't. Zero is typically not taken as a natural number in the American secondary school curriculum.) We extend **N** to **W** by adding an element 0 which satisfies

for every $x \in \mathbf{W}$, $x+0 = 0+x = x$

More formally, we extend the semigroup **N** to become a *monoid*. (A monoid is a semigroup with an identity element.)

Next, consider the integers. How can we describe the integers?

The integers have the property of additive inverses: we can subtract. But in early grades, students are told that while we can perform the operation "8-3", trying to subtract 8 from 3 is forbidden. This is because there is no natural number x so that 3-8=x. It is useful to extend the whole numbers to the integers so that we don't have such a forbidden operation.
Given any x,y in **Z** there exists a z in **Z** so that $x + z = y$. This is equivalent to saying that subtraction is defined for any two pairs of integers ($z = y-x$).

Students are typically introduced by the integers as an abstraction of a number line. Start at any arbitrary point. One can choose to move in one of two directions: right or left. +4 means move to the right by 4 and –4 means move to the left by 4.

Another way students are introduced to integers is by differences of two sets. They are given disks colored green and red, or marked with + and -, and then asked to pair up + and – disks so that there are 4 more + than – markers (to represent +4), or 4 more – than + (this represents -4), etc.

The integers are an example of a commutative *group*. A group is a monoid so that every element has an inverse. More explicitly, a group is a set S with an operation * for which we have:

- (closure) for every x,y ϵ S, we have x*y ϵ S.
- (associativity) for every x,y,z ϵ S, we have (x*y)*z = x*(y*z)
- (identity) there is an element eϵ S so that for x*e = e*x = x for every xϵ**S**
- (inverses) for every xϵS, there is an element x' so that x*x' = e = x'*x

Here is a way we can describe the integers directly in terms of **N** and +

Define the integers as a pair of natural numbers. (i.e. (m,n) for any m,n in **N**)
Using this idea, it is clear that **Z** is closed under addition. Consider the two pairs (a,b) and (c,d) where a,b,c,d in **Z**. (a,b) + (c,d) = (a + c, b + d)
Also note: (m,n) ≈ (a,b) if and only if m + b = n + a
Similarly, m – a = n – b
So, the set of integers can be "built" from pairs of natural numbers.

More specifically, **Z** = **N** x **N** / ~
The above line means that the integers are made up of pairs of natural numbers. Pairs of natural numbers are equivalent if they satisfy the relation defined by ~.
For example, consider x,y in **S**. x ~ y means that x and y satisfy some relation defined by ~. In fact, this relation is an equivalence relation.

From MAT 511 Lecture Notes: Relations
Definition: A **relation** R from X to Y is a subset of X x Y, where X is the domain of the relation and Y is the codomain of the relation.
General notation: aRb or (a,b) ϵ R or R(a,b).
If Y = X then R is a relation on X.
There exists a unique relation, called an **equivalence relation**.
In order for a relation to be an equivalence relation, it must be reflexive, symmetric and transitive.
R is reflexive if for all x in X xRx is true.
R is symmetric if for all x,y in X, xRy → yRx
R is transitive if for all x,y,z in X, xRy Λ yRz → xRz

Going from the natural numbers to the integers one can see that the numbers change from those used for counting to numbers used for comparison.

From the list at the beginning of these notes, we saw how some sets of numbers are contained in others. Using this idea, how can we "build" the set of real numbers?

Start with **N**.
N is a semigroup. Add to **N** an identity element 0 and we arrive at the whole numbers. From here, we can add the operation of subtraction to get to the set of integers. Next, division is added to achieve the rational numbers. Finally, by allowing limits we can arrive at the set of real numbers. To take this a step further, the set of complex numbers allow for the existence of solutions to all polynomials.
N → W → Z → Q → R (→ C)

The Real Numbers
The set of real numbers, as defined previously in these notes, is said to be a field with respect to addition and multiplication (Bartle & Sherbert, 22). The following nine properties are known as the field axioms. Note: a + b and a*b will be used when discussion the properties of addition and multiplication.

Algebraic Properties of **R**. On the set **R** of real numbers there are two binary operations, defined by + and * and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) a + b = b + a for all a,b in **R** (commutative property of addition);
(A2) (a + b) + c = a + (b + c) for all a,b,c in **R** (associative property of addition);
(A3) there exists an element 0 in **R** such that 0 + a = a and a + 0 = a for all a in **R** (existence of a zero element);
(A4) for each a in **R** there exists an element –a in **R** such that a + (-a) = 0 and (-a) + a = 0 (existence of negative elements);
(M1) a*b = b*a for all a,b in **R** (commutative property of multiplication);
(M2) (a*b)*c = a*(b*c) for all a,b,c in **R** (associative property of multiplication);
(M3) there exists an element 1 in **R** *distinct from* 0 such that 1*a = a and a*1 = a for all a in **R** (existence of a unit element);
(M4) for each a ≠ 0 in **R** there exists an element 1/a in **R** such that a*(1/a) = 1 and (1/a)*a = 1 (existence of reciprocals);
(D) a*(b + c) = (a*b) + (a*c) and (b + c)*a = (b*a) + (c*a) for all a,b,c in **R** (distributive property of multiplication over addition).

The first four properties are concerned with addition, the second four with multiplication and the last connects the two.
(Bartle & Sherbert)

References:

Bartle, R. & Sherbert, D. Introduction to Real Analysis, Third Edition. 2000, John

Wiley & Sons, Inc.

Viro, J. MAT 511 Lecture Notes from November 24, 2008