remark
definition

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The preimages of the critical point 0 is very tightly satisfying the condition of $< 1 - h/2$) and only one side of bow tie has the preimage. If $\Re a_3 z^3 > 0$ (I should at the unit circle around 0, but for the moment let us assume real part) then M(z)=0.

For every n, there is one preimage of 0.
The constants in the stated theorem or lemmas etc are okay but all the small bounds appearing in the proofs below have to work out carefully later.
Without loss of generality, we assume that

$$f(z) = 1 + z^2 + a_3 z^3 + O\left(z^3\right)$$

For general polynomial g with a critical point θ, we transform g to f,

$$h(z) = \frac{g((z+\theta)}{g(\theta)}$$

$$f(z) = h(\frac{z}{\sqrt{h''(0)}})$$

Often we use $F(z) = f(z)/f'(z)$

Theorem 0.1 *Let $z = x+iy$. Let z be small enough. If $3|a_3|x^2 > y > -3|a_3|x^2$ then M(z) escapes from the region and the possible bad points are in this region.*

Proof is done in the lemmas below.

Proposition 0.2 *With $|z| < r$ as above say $|\sum_{j\geq 4} a_j z^j| < 1/100 a_3 z^3$, we have*

$$\texttt{If } z \in D_r(0) \texttt{ and } |hF(z)| > c|z| \texttt{ then } \left|\frac{f(M(z))}{f(z)}\right| < 1$$

with $c < 1.23$

Proof:

$$|hF(z)| > c|z| \iff h > c|z|\left|\frac{1}{F(z)}\right| = c|z|\left|\frac{f'(z)}{f(z)}\right| > 2c|z|^2(1 - 1/100),$$

assuming that $|1/F(z)| > 2|z|9/10$

$$h > 1.8c|z|^2$$

Since

$$\left|\frac{f(M(z))}{f(z)}\right| < 1 - \frac{h}{2}$$

and

$$|f(z)| < |1 + z^2 + \epsilon| < 1 + 1.1|z|^2$$

for $|z| < r$ we have

$$|f(M(z))| < |f(z)|(1 - \frac{h}{2})| < |(1 + 1.1|z|^2)(1 - 1.1|z|^2) < 1$$

Need 1.8 c= 2.2 and $c = 2.2/1.8 > 1.23$

Remark 0.3 *The above lemma establishes the upper bound of the increment*

$$\left| h\frac{f(z)}{f'(z)} \right| > 1.23|z|$$

Without loss of generality we assume that $0 < \arg(z) < \pi/2$. Otherwise we switch the role of z.

Lemma 0.4 *If $|z| < r$ where r given as above, and $f(M(z))| > 1 = |f(0)|$ we have*

$$\textit{if } \arg(h\frac{f(z)}{f'(z)}) < 0 \textit{ then } \arg(M(z) > \arg(z)$$

and

$$\textit{if } \arg(1 - (3/2\, a_3 + \delta)z) < \arg z, \textit{ then } \arg(M(z) > \arg(z)$$

where $|\delta|$ is small and the upper bound of δ has to work on soon .

Proof:

$$M(z) = z - h\frac{f(z)}{f'(z)} = z - \frac{h}{2z}(1 - 3/2\, a_3 z + \left(1 + 9/4\, a_3{}^2\right) z^2 + O\left(z^3\right))$$

$$= z - \frac{h}{2z}(1 - (3/2\, a_3 + \delta)z),$$

where r be chosen so that for $|z| < r$, $|\delta| < |a_3/4|$ and $|(3/2\, a_3 + \delta)z| < 0.1$ Notice that

$$\text{if } \arg(h\frac{f(z)}{f'(z)}) < 0 \texttt{ and } |h\frac{f(z)}{f'(z)}| < 1.23|z| \text{ then } \arg(M(z) > \arg(z)$$

Writing out in terms of z we have

$$\frac{f(z)}{f'(z)} = (\frac{1}{z}(1 - (3/2\, a_3 + \delta)z) < 0 \iff \arg(1 - (3/2\, a_3 + \delta)z) < \arg z,$$

where $|\delta|$ is small and the upper bound of δ has to work on soon . The last condition translates into the statement in the Theorem 1.

Remark 0.5 *Together with Lemma above we have $M^n(z) \to 0$ only if $hF(z)| < 1.23|z|^2$ and $\arg(h\frac{f(z)}{f'(z)}) > 0$*

Remark 0.6 *Now by the Theorem 1, we have* $M^n(z) \to 0$ *only if* $|y| < 3|a_3|x^2$. *We call this region bow tie. Now we study the behavior of a point in the bow tie which remains in this bow tie under* $M(z)$.

Proposition 0.7 *If* $|hF(z)| < 0.65|z|$ *or almost equivalently* $h < 0.65|z|^2$ *then*

$$\left|\frac{f(M(z))}{f(z)}\right| < 1 - \frac{h}{2}$$

Proof: Recall from above that

$$\frac{f(M(z))}{f(z)} = (1-h)f(z) + hf(z)\left(\frac{f''(z)}{2!f'(z)}hF(z) + \epsilon\right) \tag{1}$$

$$\left|\frac{f''(z)}{2!f'(z)}Fh + \sum_j ()(Fh)^j\right| < \left(\frac{hf(z)}{f'(z)}) + \epsilon\right) \tag{2}$$

since for $|z| < r$, we have

$$|\frac{f''(z)}{2!f'(z)}Fh| < 0.65|z|\frac{2 + 6\,a_3 z + O\left(z\right)}{2(2\,z + 3\,a_3 z^2 + O\left(z^2\right)} < \frac{0.66}{2} < 1/3$$

and geometric sum sums to 1/2. Hence the terms in (2) sum up at most 1/2. Hence we have the claim.

Remark 0.8 *The above two propositions narrow down.*

$$0.325 < |hF(z)| < 1.23$$

and more careful study will lead to the following:
Case 1: If $0.6|z|^2$ *is a power of* $(1/2)$.

Now we study the preimages of the critical point 0.
Let us solve M(z)=0 for h

$$M(z) = z - h\frac{f}{f'} = 0$$

Below y and w are higher order terms of f and f'. Solution to M(z)=0 is

$$h = (2\,z^2 + 3\,a_3 z^3 + (w-4)\,z^4 + (-10\,a_3 - 4\,y)\,z^5 + O\left(z^6\right) \tag{3}$$

$$\frac{h}{2} = z^2 + 3/2\,a_3 z^3 + (1/2\,w - 2)\,z^4 + (-5\,a_3 - 2\,y) \tag{4}$$

$$\frac{1}{f(z)} = \left(1 + z^2 + a_3 z^3 + z^3 y\right)^{-1} \tag{5}$$

$$= (1 - z^2 + (-a_3 - y)\,z^3 + z^4 + (2\,a_3 + 2\,y)\,z^5 + O\left(z^6\right) \tag{6}$$

$$\tag{7}$$

$$|\frac{f(0}{f(z)}| < 1-\frac{h}{2} \iff |f(z)(1-\frac{h}{2}| > 1 \tag{8}$$

$$f(z)(1-\frac{h}{2}) = 1 - 1/2\, a_3 z^3 - a_4 z^4 \tag{9}$$

$$+\left(-3/2\, a_5 + 6\, a_3 a_4 - \frac{27}{8}\, {a_3}^3 - 3\, a_3 - 3/16\, a_3 \left(16\, a_4 - 18\, {a_3}^2 - 8\right) - 12\, \left(1/4\, a_4 - 1/8\right) a_3\right) z^5 \tag{10}$$

We look at the last equation to be greater than 1.

Remark 0.9 *Notice that we take M(z)=0 only if* $\Re(-1/2 a_3 z^3) < 0$. *Otherwise we have ,*

$$|1/f(z)| > (1-h/2),$$

and z will take the smaller h. $M(z) = 0$ *will be taken only in the side of bow tie which makes*

$$\Re(a_3 z) > 0$$

.

Remark 0.10 *We call the reduction condition by*

$$|\frac{f(M(z)}{f(z)}| < 1 - h/2$$

We label ξ_n *to be the solution of* $M(z) = 0$ *when* $h_n(1/2)^n$ *and satisfies the reduction condition with* h_n. *Now as we move from* ξ_n *to the left and if it is not in the good set then* $|f(z)|$ *increases. With* $h = (1/2)^n$ *the reduction condition satisfies for a while until it reaches the point* ω_{n+1}, *such that*

$$h \sim |z^2| \sim< (1/2)^{n+1}$$

Notice that right side of this point we have $(1/2)^n f(z)/f'(z) < 1.23|z|$ *by the Proposition above which makes* $M(z)$ *lie in the right side of the bow tie but inside of the circle of radius* $0.23|z|$. *The left side of* ω_{n+1} *the h is* $(1/2)^{n+1}$ *and*

$$(1/2)^{n+1} f(z)/f'(z) \sim< \frac{1.23}{2}|z| = 0.615|z|$$

which satisfies the reduction condition. Hence we establish that left closed set to ω_{n+1} *uses* $(1/2)^{n+1}$ *and right open set to the* ω_1 *uses* $(1/2)^n$
On the other hand as we move to the right hand side of ξ_n *again* $|f(z)|$ *is increasing hence the reduction condition satisfies.*
Using the above argument starting from ξ_{n-1} *it will travel no further than the point*

$$(1/2)^n f(z)/f'(z) \sim< 0.615|z|$$

.

We have

$$|\omega_n| \sim \sqrt{2}\omega_{n+1} \text{ and } \xi_n \sim \sqrt{2}\xi_{n+1}$$

Hence the thickness of each strip using the same h is

$$(1 - 1\sqrt{2})|\xi_n|^2 \sim 0.292|\xi|^2$$

Further the image of this strip lies $-.23|\omega_{n+1}|$ *and* $0.4|\omega_n|$