ON BRILLOUIN ZONES

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Abstract

Given a metric space (X, d), a discrete set $S \subset X$, and a preferred point $x_0 \in S$, the n-th Brillouin zone is defined (loosely speaking) as those points in X which are closest to exactly n elements of S, including x_0. This notion is a direct generalization of the concept introduced by Brillouin [Br] in the thirties, which describes some quantum mechanical properties of crystals.

We generalize a theorem of Bieberbach [Bi] asserting that the Brillouin zones tile the underlying space, and that each zone has the same area. We then use these ideas to discuss focusing of geodesics in orbifolds of constant curvature. In the particular case of the Riemann surfaces $\mathbb{H}^2/\Gamma(N)$ ($N = 2, 3,$ or 5), we count the number of geodesics of length t that connect the origin to itself.

1 Introduction

In solid-state physics, the notion of Brillouin zones is used to describe quantum mechanical properties of a crystal. In a crystal, the atoms are arranged in a lattice; for example, in NaCl, the sodium and chorine atoms are arranged along the points of the simple cubic lattice $\mathbb{Z}^3$. If we pick a specific atom and call it the origin, its *first Brillouin zone* consists of the points in $\mathbb{R}^3$ which are closer to the origin than to any other element of the lattice. This same zone can be constructed as follows: for each element a in the lattice, let L_{0a} be the perpendicular bisecting plane of the line between 0 and a (this plane is called a Bragg plane). The volume about the origin enclosed by these intersecting planes is the first Brillouin zone, $b_1(0)$. This construction also allows us to define the higher Brillouin zones as well: a point x is in b_n if the line connecting it to the origin crosses exactly $n-1$ planes L_{0a}, counted with multiplicity.

This notion was introduced by Brillouin in the 1930s ([Br]), and plays an important role in solid-state theory (see, for example, [AM], or [Jo2], also [Ti]). The construction which gives rise to Brillouin zones is not limited to consideration of crystals, however. For example, in computational geometry, the notion of the Voronoi cell corresponds exactly to the first Brillouin zone described above (see [PS]). We shall also see below how, after suitable generalization, this construction coincides with the Dirichlet domain of Riemannian geometry, and in many cases, to the focal decomposition introduced in [Pe1] (see also [Pe3]).

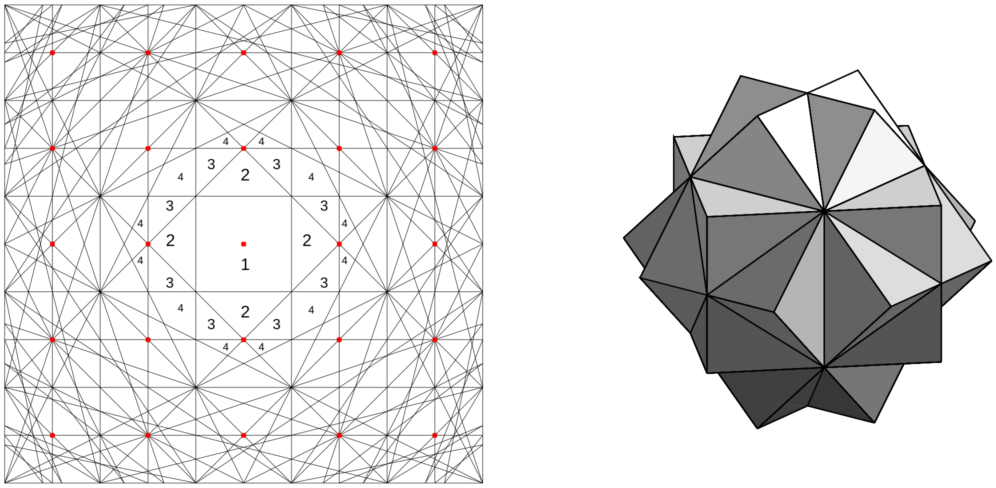

Figure 1.1: On the right are the Brillouin zones for the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. On the left is the outer boundary of the third Brillouin zone for the lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$.

With some slight hypotheses (see section 2), we generalize the construction of Brillouin zones to any discrete set S in a path-connected, proper metric space X. We generalize the Bragg planes above as mediatrices, defined here.

Definition 1.1 *For a and b distinct points in S, define the* mediatrix *(also called the equidistant set or bisector) L_{ab} of a and b as:*

$$L_{ab} = \{x \in X \mid d(x,a) = d(x,b)\}\,.$$

Now choose a preferred point x_0 in S, and consider the collection of mediatrices $\{L_{x_0,s}\}_{s\in S}$. These partition X into Brillouin zones as above: roughly, the n-th Brillouin zone $B_n(x_0)$ consists of those points in X which are accessible from x_0 by crossing exactly $n-1$ mediatrices. (There is some difficulty accounting for multiple crossings— see definition 2.7 for a precise statement.)

One basic property of the zones B_n is that they tile the space X:

$$\bigcup_{x_i\in S} B_n(x_i) = X \qquad \text{and} \qquad B_n(x_0)\cap B_n(x_1) \text{ is small}\,.$$

Here, with some extra hypotheses, "small" means of measure zero. Furthermore, again with some extra hypothesis, each zone B_n has the same area. (This property was "obvious" to Brillouin). Both results were proved by Bieberbach in [Bi] in the case of a lattice in $\mathbb{R}^2$. Indeed, he proves (as we do) that each zone forms a fundamental set for the group action of the lattice. His arguments rely heavily on planar Euclidean geometry, although he remarks that his considerations work equally well in $\mathbb{R}^d$ and can be extended to "groups of motions in non-Euclidean spaces". In [Jo1], Jones proves these results for lattices in $\mathbb{R}^d$, as well as giving asymptotics for both the distance from B_n to the basepoint, and for the number of connected components of the interior of B_n. In section 2, we show that the tiling result holds for arbitrary discrete sets in a metric space. If the discrete set is generated by a group of isometries, we show that each B_n forms a fundamental set, and consequently all have the same area (see Prop. 2.11).

We now discuss the relationship of Brillouin zones and focal decomposition of Riemannian manifolds.

If $x_1(t)$ and $x_2(t)$ are two solutions of a second order differential equation with $x_1(0) = x_2(0)$ and there is some $T \neq 0$ so that $x_1(T) = x_2(T)$, then the trajectories x_1 and x_2 are said to *focus* at time T. One can ask how the number of trajectories which focus varies with the endpoint $x(T)$— this gives rise to the concept of a *focal decomposition* (originally called a *sigma decomposition*). This concept was introduced in [Pe1] and has important applications in physics, for example when computing the semiclassical quantization using the Feynman path integral method (see [Pe3]). There is also a connection with the arithmetic of positive definite quadratic forms (see [Pe2], [KP], and [Pe3]). Brillouin zones have a similar connection with arithmetic, as can be seen in section 4.

More specifically, consider the two-point boundary problem

$$\ddot{x} = f(t, x, \dot{x}), \qquad x(t_0) = x_0, \qquad x(t_1) = x_1, \qquad x, t, \dot{x}, \ddot{x} \in \mathbb{R}.$$

Associated with this equation, there is a partition of $\mathbb{R}^4$ into sets Σ_k, where (x_0, x_1, t_0, t_1) is in Σ_k if there are exactly k solutions which connect (x_0, t_0) to (x_1, t_1). This partition is the focal decomposition with respect to the boundary value problem. In [PT], several explicit examples are worked out, in particular the fundamental example of the pendulum $\ddot{x} = -\sin x$. Also, using results of Hironaka ([Hi]) and Hardt ([Ha]), the possibility of a general, analytic theory was pointed out. In particular, under very general hypotheses, the focal decomposition yields an analytic Whitney stratification.

Later, in [KP], the idea of focal decomposition was approached in the context of geodesics of a Riemannian manifold M (in addition to a reformulation of the main theorem of [PT]). Here, one takes a basepoint x_0 in the manifold M: two geodesics γ_1 and γ_2 *focus* at some point $y \in M$ if $\gamma_1(T) = y = \gamma_2(T)$. This gives rise to a decomposition of the tangent space of M at x into regions where the same number of geodesics focus.

In order to study focusing of geodesics on an orbifold (M, g) with metric g via Brillouin zones, we do the following. Choose a base-point p_0 in M and construct the universal cover X, lifting p_0 to a point x_0 in X. Let γ be a smooth curve in M with initial point p_0 and endpoint p. Lift γ to $\tilde{\gamma}$ in X with initial point x_0. Its endpoint will be some $x \in \pi^{-1}(p)$. The metric g on M is lifted to a metric $\tilde{g}$ on X by setting $\tilde{g} = \pi^* g$. Under the above conditions, the group G of deck transformations is discontinuous and so $\pi^{-1}(p_0) \subset X$ is a discrete set. One can ask how many geodesics of length t there are which start at p_0 end in p, or translated to $(X, \tilde{\gamma})$, this becomes: *How many mediatrices $L_{x_0,s}$ intersect at x, as s ranges over $\pi^{-1}(p_0)$?*

Notice that if the universal cover of M coincides with the tangent space TM_x, the focal decomposition of [KP] and that given by Brillouin zones will be the same. If the universal cover and the tangent space are homeomorphic (as is the case for a manifold of constant negative curvature), the two decompositions are not identical, but there is a clear correspondence. However, if the universal cover of the manifold is not homeomorphic to the tangent space at the base point, the focal decomposition and that given by constructing Brillouin zones in the universal cover are completely different. For example, let M be $\mathbb{S}^n$, and let x be any point in it. The focal decomposition with respect to x gives a collection of nested $n-1$-spheres centered at x; on each of these infinitely many geodesics focus (each sphere is mapped by the exponential to either x or its antipodal point). Between the spheres are bands in which no focusing occurs. (See [Pe3]). However, using the construction outlined in the previous paragraph gives a very different result. Since $\mathbb{S}^n$ is simply connected, it is its own universal cover. There is only one point in our discrete set, and so the entire sphere S^n is in the first zone B_1.

The organization of this paper is as follows. In section 2, we set up the general machinery we need, and prove the main theorems in the context of a discrete set S in a proper metric space.

Section 3 explores this in the context of manifolds of constant curvature. The universal cover is $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, and the group G of deck transformations is a discrete group of isometries (see [doC]). The discrete set S is the orbit of a point not fixed by any element of G under this discontinuous group. It is easy to see that the mediatrices in this case are totally geodesic spaces. From the basic property explained above, one can deduce that the n-th Brillouin zone is a fundamental domain for the group G in X.

In section 4, we calculate exactly the number of geodesics of length t that connect the origin to itself in two cases: the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ and the Riemann surfaces $\mathbb{H}^2/\Gamma(p)$, for $p \in \{2, 3, 5\}$. While these calculations could, of course, be done independent of our construction, the Brillouin zones help visualize the process.

In the final section, we give a nontrivial example in the case of a non-Riemannian metric, and mention a connection to the question of how many integer solutions there are to the equation $a^k + b^k = n$, for fixed k.

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2 Definitions and main results

In this section, we prove that under very general conditions, Brillouin zones tile (as defined below) the space in which they are defined, generalizing an old result of Bieberbach [Bi]. With stronger assumptions, we prove that these tiles are in fact well-behaved sets: they are equal to the closure of their interior.

Notation: Throughout this paper, we shall assume X is a path connected, proper (see below) metric space (with metric $d(\cdot,\cdot)$). We will make use the following notation:

- Write an open r-neighborhood of a point x_0 as $N_r(x_0) = \{x \in X \mid d(x_0, x) < r\}$.
- Define the circumference as $C_r(x_0) = \{x \in X \mid d(x_0, x) = r\}$.
- Their union is the closed disk of radius r, denoted by $D_r(x_0) = \{x \in X \mid d(x_0, x) \leq r\}$.

Definition 2.1 *A metric space X is* proper *if the distance function $d(x,\cdot)$ is a proper map for every fixed $x \in X$. In particular, for every $x \in X$ and $r > 0$, the closed ball $D_r(x)$ is compact. Such a metric space is also sometimes called a* geometry *(See [Ca]).*

Note if X is proper, it is locally compact and complete. The converse also holds if X is a geodesic metric space (see Thm. 1.10 of [Gr]). The metric spaces considered here need not be geodesic.

Definition 2.2 *The space X is called* metrically consistent *if, for all x in X, $R > r > 0$ in $\mathbb{R}$, and for each $a \in C_R(x)$, there is a $z \in C_r(x)$ satisfying $N_{d(z,a)}(z) \subseteq N_R(x)$ and $C_{d(z,a)}(z) \cap C_R(x) = \{a\}$.*

This property is satisfied for any Riemannian metric.

Note that any mediatrix $L_{a,b}$ separates X, that is: $X - L_{ab}$ contains at least two components (one containing the point a and the other b).

Definition 2.3 *We say that the mediatrix L_{ab} is* minimally separating *if for any subset $\tilde{L} \subset L_{ab}$ with $\tilde{L} \neq L_{ab}$, the set $X - \tilde{L}$ has one component.*

Let L_{0a} be a mediatrix. If it is minimally separating, then $X - L$ has exactly two components. Also note that if a separating set $L \subseteq X$ contains a non-empty open set V, it cannot be minimally separating. For if A and B are disjoint open sets containing $X - L$, then so are $\tilde{A} = A - (A \cap V)$ and $\tilde{B} = B - (B \cap V)$. Now let x be any point in V. Then it is easy to see that the disjoint open sets $\tilde{A} \cup V$ and $\tilde{B}$ cover $X - (L - x)$. Thus $L - x$ separates.

We define the following sets

$$L_{0a}^- = \{x \in X \mid d(0,x) - d(a,x) < 0\}, \quad L_{0a}^+ = \{x \in X \mid d(0,x) - d(a,x) > 0\}.$$

Definition 2.4 *Two minimally separating sets L_{0a} and L_{0b} are* topologically transversal *if they are disjoint, or if for each $x \in L_{0a} \cap L_{0b}$ and every neighborhood V of x, the sets $L_{0a}^+ \cap L_{0b}^+ \cap V$, $L_{0a}^+ \cap L_{0b}^- \cap V$, $L_{0a}^- \cap L_{0b}^+ \cap V$, $L_{0a}^- \cap L_{0b}^- \cap V$ are all nonempty.*

Usually, there will be a discrete set of points $S = \{x_i\}_{i \in I}$ in X which will be of interest. By discrete we mean that any compact subset of X contains finitely many points of S. Note that if $\liminf\limits_{a,b \in S} d(a,b) > 0$, then S is discrete.

Definition 2.5 *We say a proper, path connected metric space X is* Brillouin *if it satisfies the following conditions:*

1: X is metrically consistent.

2: For all a, b in X, the mediatrices L_{ab} are minimally separating sets.

3: For any three distinct points 0, a, and b in X, the mediatrices L_{0a} and L_{0b} are topologically transversal.

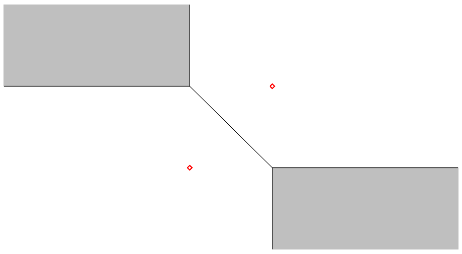

Figure 2.1: The set $L_{(0,0),(a,a)}$ contains two quarter-planes.

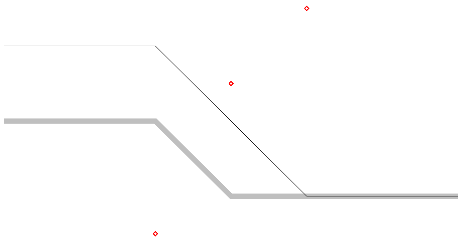

Figure 2.2: $L_{(0,0),(4,6)}$ (thin solid line) and $L_{(0,0),(2,4)}$ (thick grey line) are not transverse.

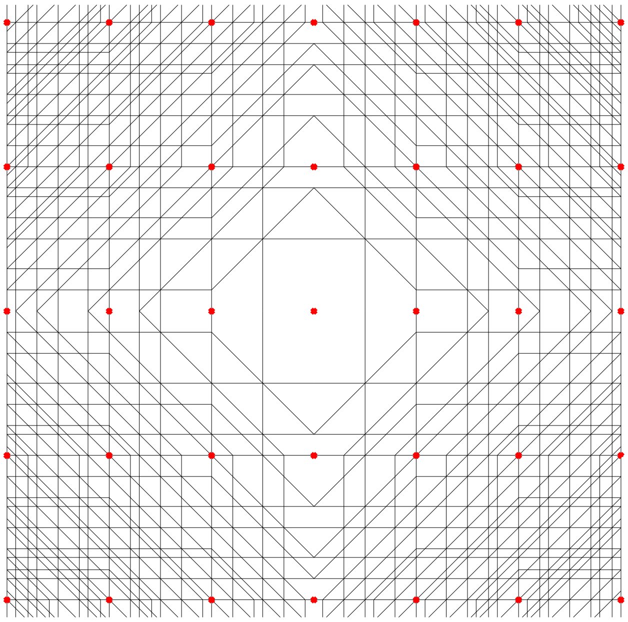

Figure 2.3: The mediatrices L_{0a} for $\mathbb{R}^2$ with the Manhattan metric and a in the lattice $\{(m, n\sqrt{2})\}$.

The last two conditions in the above definition may be weakened to apply only to those mediatrices L_{ab} where a and b in S. In this case, we will say that X *is Brillouin over* S, if it is not obvious from the context.

Example 2.6 Equip $\mathbb{R}^2$ with the "Manhattan metric", that is, $d(p,q) = |p_1 - q_1| + |p_2 - q_2|$. In this metric, a circle $C_r(p)$ is a diamond of side length $r\sqrt{2}$ centered at p, so condition 1 is satisfied. However, condition 2 fails: if the coordinates of a point a are equal, then L_{0a} consists of a line segment and two quarter-planes (see Fig. 2.1). If the discrete set S contains no such points, we can still run into trouble with topological transversality. For example, the mediatrices $L_{(0,0),(2,4)}$ and $L_{(0,0),(4,6)}$ both contain the ray $\{(t,1) \mid t \geq 4\}$ (Fig. 2.2). But, if we are careful, we can ensure that the space is Brillouin over S. To achieve this, we must have that for all pairs (a_1, a_2) and (b_1, b_2) in S, $a_1 - a_2 \neq b_1 - b_2$. For example, take S to be an irrational lattice such as $\left\{(m, n\sqrt{2}) \,\middle|\, m, n \in \mathbb{Z}\right\}$. It is easy to check that for all $0, a, b \in S$, the properties of definition 2.5 are true. (From this example, we see that to do well in Manhattan, one should be carefully irrational.)

As mentioned in the introduction, for each $x_0 \in S$, the mediatrices $L_{x_0 a}$ give a partition of X. Informally, those elements of the partition which are reached by crossing $n-1$ mediatrices from x_0 form the n-th Brillouin zone, $B_n(x_0)$. This definition is impractical, in part because a path may cross several mediatrices simultaneously, or the same mediatrix more than once. Instead, we will use a definition given in terms of the number of elements of S which are nearest to x. This definition is equivalent to the informal one when X is Brillouin over S (see Prop. 2.13 below). We use the notation $\#(S)$ to denote the cardinality of the set S.

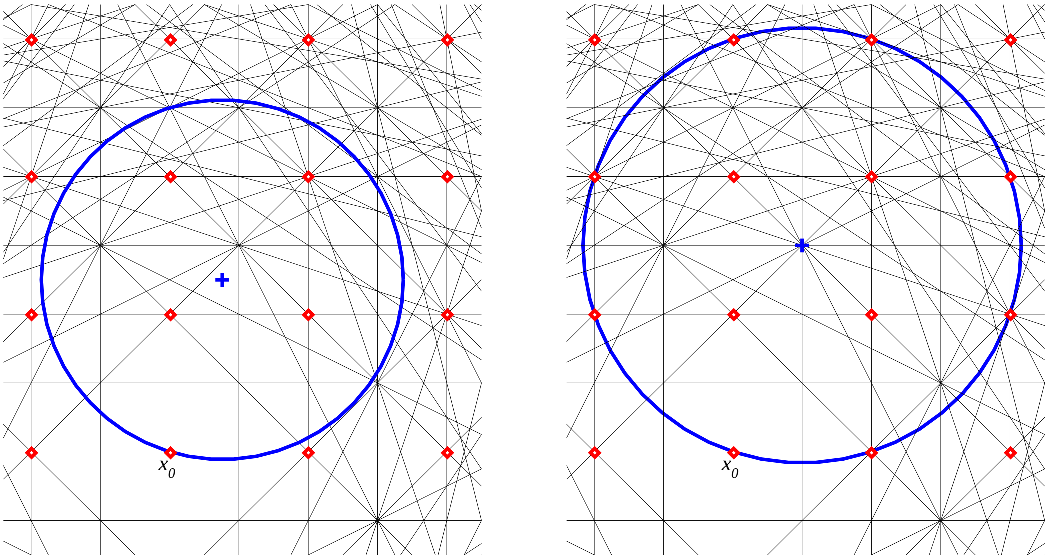

Figure 2.4: Here we illustrate the definition of the sets $b_n(x_0)$ and $B_n(x_0)$ for the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. In both pictures, the circle $C_{d(x,x_0)}(x)$ is drawn, and the basepoint x_0 lies in the center of the square at the lower left. On the left side, the point x (marked by a small cross) lies in b_5, and $\#(N_r(x) \cap S) = 4$, while x_0 is the only point of S on the circle. On the right, we have $m = 4$ and $\ell = 8$, so x lies in all of the sets $B_5, B_6, \ldots, B_{12}$.

Definition 2.7 *Let $x \in X$, let n be a positive integer, $n \leq \#(S)$, and let $r = d(x, x_0)$. Then define the sets $b_n(x_0)$ and $B_n(x_0)$ as follows.*

- $x \in b_n(x_0) \iff \#(N_r(x) \cap S) = n - 1$ *and* $C_r(x) \cap S = \{x_0\}$.
- $x \in B_n(x_0) \iff \#(N_r(x) \cap S) = m$ *and* $\#(C_r(x) \cap S) = \ell$, *where* $l, m \in \mathbb{Z}^+$ *with* $m + 1 \leq n \leq m + \ell$.

Here the point x_0 is called the *base point*, and the set $B_n(x_0)$ is the n-th Brillouin zone with base point x_0. Note that in the second part, if $m = n - 1$ and $\ell = 1$, then $x \in b_n(x_0)$. So $b_n(x_0) \subseteq B_n(x_0)$. Note also that the complement of $b_n(x_0)$ in $B_n(x_0)$ consists of subsets of mediatrices (see Def. 1.1). Note also that $b_n(x_0)$ is open and that $B_n(x_0)$ is closed. Finally, observe that for fixed x_0 the sets $b_n(x_0)$ are disjoint, but the sets $B_n(x_0)$ are not. In what follows it will be proved that $\{B_n(x_0)\}_{n>0}$ cover the space X. Thus we can assign to each point x its Brillouin index as the the largest n for which $x \in B_n(x_0)$. This definition was first given in [Pe1].

The following lemma, which follows immediately from Def. 2.7, explains a basic feature of the zones, namely that they are concentric in in a weak sense. This property is also apparent from the figures.

Lemma 2.8 *Any continuous path from x_0 to $B_n(x_0)$ intersects $B_{n-1}(x_0)$.*

The Brillouin zones actually form a covering of X by non-overlapping closed sets in various ways. This is proved in parts. The next two results assert that the zones B cover X, but the zones b do not. The first of these is an immediate consequence of the definitions. The second is more surprising and leads to corollary 3.4. The fact that the $B_i(x_n)$ is the closure of $b_i(x_n)$ (and thus that the interiors do not overlap) is proved in proposition 2.12. We will use the word "tiling" for a covering by non-overlapping closed sets.

Lemma 2.9 *For fixed n the Brillouin zones tile X in the following sense:*

$$\bigcup_i B_i(x_n) = X \quad \text{and} \quad b_i(x_n) \cap b_j(x_n) = \emptyset \quad \text{if } i \neq j.$$

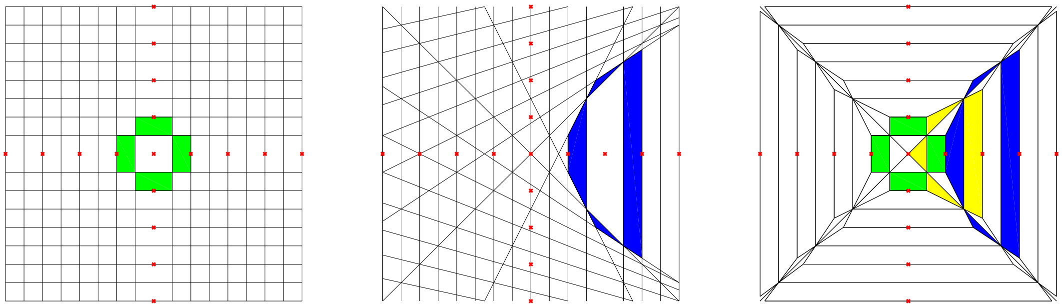

Figure 2.5: This example illustrates Lemma 2.9 and Thm. 2.10. Let S be the discrete set $\{(m,0)\} \cup \{(0,n)\}, m,n \in \mathbb{Z}$ in the Euclidean plane. On the left is the tiling given by $B_i(0,0)$ and in the middle is the tiling by $B_i(2,0)$. In both cases, b_2 is shaded. On the right is the tiling given by $B_2(x_i)$ as in Thm. 2.10. The sets $b_2(0,0)$, $b_2(1,0)$, and $b_2(2,0)$ have been shaded. Note that this S does not correspond to a group, nor does it satisfy the hypotheses of Prop. 2.11, because there are no isometries which permute S and do not fix the origin.

Theorem 2.10 *Let X be a proper, path connected metric space let $S = \{x_i\}_{i\in I}$ be a discrete set. Then, for fixed $n \leq \#(S)$, the sets $\{B_{n_0}(x_i)\}_{i\in I}$ tile X in the following sense:*

$$\bigcup_i B_n(x_i) = X \quad \text{and} \quad b_n(x_i) \cap b_n(x_j) = \emptyset \quad \text{if } i \neq j.$$

Proof: First, we show that for any fixed $n > 0$ and each $x \in X$, there is an $x_i \in S$ with $x \in B_n(x_i)$. Re-index S so that if $S = \{x_1, x_2, x_3, \ldots\}$ and $i < j$, then $d(x, x_i) \leq d(x, x_j)$. This can be done; since S is a discrete subset and closed balls $D_c(x_i)$ are compact, the subsets of S with $d(x, x_i) \leq c$ are all finite. Let $r_i = d(x, x_i)$. We will show that $x \in B_n(x_n)$.

Note that $r_n \geq r_{n-1}$. Suppose first that $r_n > r_{n-1}$, then $N_{r_n}(x) \cap S$ contains exactly $n-1$ points, and $x_n \in C_{r_n}(x) \cap S$. Thus $x \in B_n(x_n)$. Note that if $r_{n+1} > r_n$, then we would have $x \in b_n(x_n) \subset B_n(x_n)$.

If, on the other hand, $r_n = r_{n-1}$, then there is a $k > 0$ so that $r_n = r_{n-1} = \ldots = r_{n-k}$, and so $\#(N_{r_n}(x) \cap S) = n - k - 1 \leq n - 1$. But then $\#(C_{r_n}(x) \cap S) \geq k + 1$, and hence $x \in B_n(x_n)$ as desired.

For the second part, we show that $b_n(x_i) \cap b_n(x_j) = \emptyset$. If not, then there is a point x in their intersection. If $r_i = r_j$, then $x_i = x_j$, because by the definition of $b_n(x_k)$, $\{x_k\} = C_{r_k}(x) \cap S$. If not, then $r_i < r_j$. In this case, $x_i \in D_{r_i}(x) \subset N_{r_j}(x)$. Thus, since $\#(N_{r_i}(x) \cap S) = n - 1$, $N_{r_j}(x)$ must contain at least n points of S, a contradiction. □

The next result indicates how this notion of tiling is related to the notion of fundamental domain.

Proposition 2.11 *Let S be a discrete set in a metric space X as in theorem 2.10. Suppose that for each x_i in S there is an isometry g_i of X such that $g_i(x_0) = x_i$, g_i permutes S and the only g_i which leaves x_0 fixed is the identity. Then there is a set F (the fundamental set), satisfying:*

$$b_n(x_0) \subseteq \quad F \quad \subseteq B_n(x_0) \quad \text{with}$$

$$\bigcup_i g_i(F) = X \quad \text{and} \quad g_i(F) \cap g_j(F) = \emptyset \ (i \neq j).$$

Proof: Suppose that $x \in b_n(x_0)$. From definition 2.7 and the fact that the g_i are isometries, we see that this is equivalent to $g_i(x) \in b_n(x_i)$. Thus $g_i(b_n(x_0)) = b_n(x_i)$. Now apply theorem 2.10. A similar reasoning proves the statement for $B_n(x_0)$. □

Remark: The fundamental set F is not necessarily connected. Also, note that it follows from this proposition that $B_i(x_0)$ is scissors congruent to $B_j(x_0)$ (see [Sah] for a discussion of scissors congruence). In particular, this implies immediately that the B_i all have the same area. Note that this result does not hold if S is not generated by a group of isometries. For example, the "face-centered cubic" structures common to many crystals do not satisfy this, and the corresponding B_n have different volumes, although they still tile the space. See also figure 2.5.

The preceding results are valid without the transversality conditions on the metric, as the reader can check. We will now use the additional properties of the metric to derive a regularity result.

Proposition 2.12 *If X is Brillouin (over S), then $B_n(x_i)$ is the closure of $b_n(x_i)$.*

Proof: Let $x \in B_n(x_0)$. If $x \in b_n(x_0)$, we are done. So suppose $x \notin b_n(x_0)$, and then by Def. 2.7 (with $r = d(x, x_0)$), $N_r(x)$ contains $m \leq n-1$ points of S and $C_r(x)$ contains $\ell \geq n-m$ with $\ell \geq 2$ points of S, including x_0. Thus x lies in the intersection of $\ell - 1 \geq 1$ mediatrices $\{L_{x_0,x_i}\}_{i=1}^{\ell-1}$.

Suppose $\ell = 2$, so x lies on a single mediatrix L_1. Then by condition 2, x is in the closure of L_1^- and L_1^+. Thus if $x' \in L_1^-$ and close enough to x, then $x' \in b_{m+1}(x_0)$. If $x' \in L_1^+$ and sufficiently close to x, then $x' \in b_{m+2}(x_0)$. From our choices, we see that $m = n-1$ or $m = n-2$. In either case, we are done.

The proof for $\ell > 2$ is slightly more involved (we have to use transversality), but the idea is the same. The point x lies in the intersection of at least two mediatrices. Essentially, we need show that the mediatrices disentangle in every neighborhood of x. We will prove that for all $j \in \{1, \ldots, \ell\}$, $x \in \overline{b_{m+j}(x_0)}$. Note that $n - \ell \leq m \leq n-1$, and so showing this is sufficient.

Again, since mediatrices are minimally separating, we have that for all $i \in \{1, \ldots, \ell - 1\}$ there exist open sets L_i^- and L_i^+ in any neighborhood of x for which

$$d(x, x_0) < d(x, x_i) \text{ for } x \in L_i^- \quad \text{and} \quad d(x, x_0) > d(x, x_i) \text{ for } x \in L_i^+.$$

Noting that $x_i \in C_r(x)$, then from condition 1 of Def. 2.7, we have that for any small $\rho > 0$, there is a point $z^- \in C_\rho(x)$ for which $d(z^-, x_0) < d(z^-, x_i)$ for all $i \in \{1, \dots, \ell\}$. Thus $\bigcap_i L_i^-$ is nonempty, and is contained in $b_{m+1}(x_0)$. By the same principle, we can choose a point $z^+ \in C_r(x)$ for which $d(z^+, x_0) > d(z^+, x_i)$. As a consequence, $\bigcap_i L_i^+$ is nonempty, and is a subset of $b_{m+\ell}(x_0)$.

Now, $b_{m+1}(x_0)$ and $b_{m+\ell}(x_0)$ are disjoint. If $\ell > 1$, their complement in a neighborhood of x must contain at least two mediatrices. By transversality (condition 3), these cannot coincide. Thus V must be intersected by some $b_{m+i}(x_0)$ for $i \in \{1, \dots \ell\}$. By induction one finishes the argument. □

Remark: The references to mediatrices in the above proof may make it appear that we are relying on the informal definition of B_n rather than Definition 2.7. In fact, we are only using the fact that $B_n \setminus b_n$ consists of subsets of mediatrices, which was noted following Def. 2.7. In fact, for Brillouin spaces, the informal definition of b_n by mediatrices and Definition 2.7 are equivalent, as the following proposition shows.

Proposition 2.13 *Let X be Brillouin over S. Then there is a path γ from x to x_0 which crosses exactly $n-1$ mediatrices with no multiple crossings if and only if $x \in b_n(x_0)$. By "no multiple crossings" we mean that if L_i and L_j are distinct mediatrices, γ intersects L_i in at most a single point, and $\gamma \cap L_i \neq \gamma \cap L_j$.*

Proof: First, suppose there is such a path γ from x_0 to x. As long as $\gamma(t)$ crosses no mediatrices, the number of elements of S contained in $D_r(\gamma(t))$ remains constant. If this number is i, then $\gamma(t) \in b_i(x_0)$. Each time $\gamma(t)$ crosses a single mediatrix, this number increases by 1. Thus, after crossing $n-1$ mediatrices, $\gamma(t)$ will be in $b_n(x_0)$.

Now, let $x \in b_n(x_0)$. We must construct a path from x to x_0 which crosses exactly $n-1$ mediatrices. Let b_n^* denote the connected component of $b_n(x_0)$ which contains x, and choose a point y in $B_{n-1}(x_0) \cap \partial b_n^*$. Such a point always exists by Lemma 2.8 and Prop. 2.12. In fact, because ∂b_n consists of subsets of mediatrices, topological transversality of mediatrices allows us to choose y so that it lies on a single mediatrix.

There certainly is a path connecting x to y which crosses no mediatrices; by similar reasoning, we can choose $x' \in b_{n-1}(x_0)$ and a path from x' to y which crosses no mediatrices. We have now constructed a path from $x \in b_n$ to $x' \in b_{n-1}$ which crosses exactly one mediatrix—repeating the argument $n-2$ times finishes the proof. □

3 Brillouin zones in spaces of constant curvature

In this section X will be assumed to be one of $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, all equipped with the standard metric, and let G be a discontinuous group of isometries of X. Denote the quotient X/G with the induced metric by (M, g). Then the construction of lifting to the universal cover, as outlined in the introduction, applies naturally to (M, g). In this section we describe focusing of geodesics in (M, g) by Brillouin zones in X. The discrete set S is given by the orbit of a chosen point in

X (which we will call the origin) under the group of deck-transformations G. The fact that the Brillouin zones are fundamental domains is now a direct corollary of proposition 2.11.

The regularity conditions of Def. 2.5 are easily verified in the present context. We do this first.

Lemma 3.1 *If X is either $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, then a mediatrix L_{ab} in X is an $(n-1)$-dimensional, totally geodesic subspace consisting of one component, and $X - L_{ab}$ has two components.*

Proof: This is easy to see if we change coordinates by an isometry of X, putting a and b in a convenient position, say as x and $-x$. The mediatrix $L_{x,-x}$ is easily seen to satisfy the conditions (in the case of $\mathbb{S}^n$, it is the equator, and for the others, it is a hyperplane). The conclusion follows. □

Proposition 3.2 *All such spaces X are Brillouin (see definition 2.5).*

Proof: As remarked before, the first condition is satisfied for any Riemannian metric. The second condition is also easy. It suffices to observe that the subspaces of Lemma 3.1 are minimally separating.

To prove the topological transversality condition, note that lemma 3.1 implies that if L_{0a} and L_{0b} coincide in some open set, then they are equal. One easily sees that then we must have $L_{0a} = L_{0b} = L_{ab} = L$. Applying the second part of the lemma to L_{0a}, L_{0b}, and L_{ab}, we see that they must separate X in at least 3 components, namely one for each point a, b, and 0. This leads to a contradiction. □

Remark: Note that in fact, the mediatrices are transversal in the usual sense. If L_{0a} and L_{0b} coincide in an open set, then their tangent spaces also coincide at some point. Uniqueness of solutions of second order differential equations then implies $L_{0a} = L_{0b}$.

Recall that a metric space X is called *rigid* if the only isometry which fixes each point of a nonempty open subset of X is the identity. It is not hard to see that $\mathbb{S}^n$, $\mathbb{H}^n$, and $\mathbb{R}^n$ are rigid spaces. See [Ra] for more details of rigid metric spaces and for the proof of the following result. Recall that the stabilizer in G of a point $x \in X$ consists of those elements of G that fix x.

Proposition 3.3 *Let G be a discontinuous group of isometries of a rigid metric space X. Then there exists a point y of X whose stabilizer G_y is consists of the identity.*

We now return to Brillouin zones as defined in the last section. Recall that G is a group of isometries of X that acts discontinuously on points in X. Let x_0 be a point in X whose stabilizer under the action of G is trivial. Let $x \in X$ be a point in the orbit of x_0 under G, that is to say, $x = \gamma(x_0)$ for some $\gamma \in G - \{e\}$, where e is the identity of G. Let $[x_0, x]$ be a geodesic segment of minimal length whose endpoints are x_0 and x. Then $B_n(x_0)$, the n-th Brillouin zone relative to x_0, is the set of points in X such that the segment $[x_0, x]$ intercepts exactly $n-1$ mediatrices $L_{x_0,y}$, where y is in the orbit of x_0 under the group G. Proposition 2.11 immediately implies the most important fact about Brillouin zones in this setting.

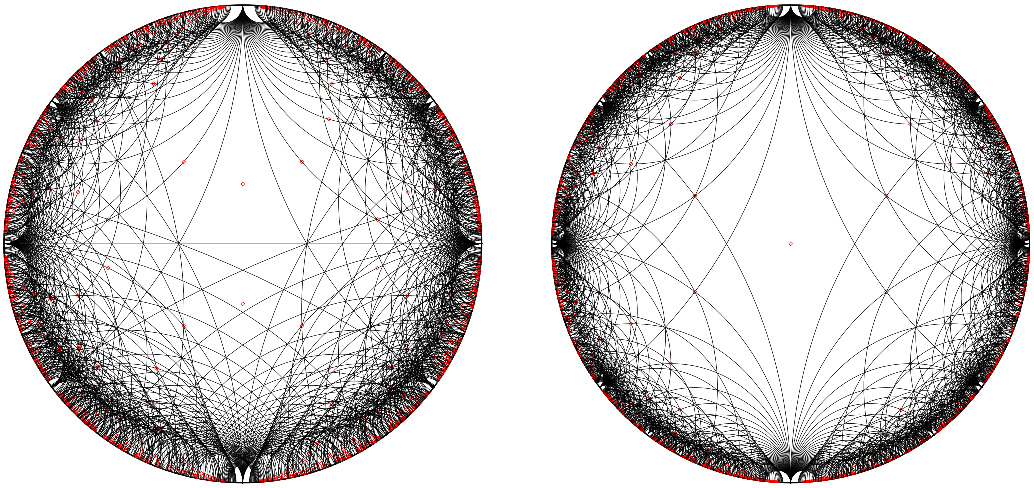

Figure 3.1: The Brillouin zones for $PSL(2,\mathbb{Z})$ in the hyperbolic disk. We have transported the "usual" upper half-plane representation using the map $z \mapsto \frac{iz+1}{z+i}$. On the left is shown the sets $B_n(\frac{i}{4})$, which give fundamental domains as in Cor. 3.4. On the right, 0 is taken as a basepoint. Since the origin has a non-trivial stabilizer, the corresponding Brillouin zones give a double cover of the fundamental domains.

Corollary 3.4 *Let X be $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, and let G be a discontinuous group of isometries of X. Let $x_0 \in X$ be such that its stabilizer G_{x_0} under G is trivial. Then for every positive integer n, the n-th Brillouin zone $B_n(x_0)$ is a fundamental domain for the action of G on points in X. Its boundary is the union of pieces of totally geodesic subspaces and equals the boundary of its interior.*

Remark: The first Brillouin zone $B_1(0)$ is the usual Dirichlet fundamental domain for the action of G. Furthermore, even when G_{x_0} is not trivial, $B_n(x_0)$ is a k-fold cover of a fundamental domain.

As pointed out in the introduction, the number of geodesics that focus in a certain point is counted in the lift. So if a given point $x \in X$ is intersected by n mediatrices, it is reached by $n+1$ geodesics of length $d(0,x)$ emanating from the reference point (the origin). In the next section, we give more specific examples of this.

Finally, we state a conjecture.

Conjecture 3.5 *Let $(X,\tilde{g})$ be the universal cover of a d-dimensional smooth Riemannian manifold (M,g) as described in the construction. For a generic metric g on M, no more than d mediatrices intersect in any given point y of X.*

This conjecture acquires perhaps even more interest (and certainly more structure), when one restricts the collection of metrics on M to conformal ones ([Mas]). A result in this direction for $M = \mathbb{R}^2/\mathbb{Z}^2$ can be found in [Jo1].

4 Focusing in two Riemannian examples

In this section, we give two examples (one of them new as far as we know) of focusing. Suppose that at $t = 0$ geodesics start emanating in all possible directions from the origin. At certain times $t_1, t_2, ...$, we will see geodesics returning to the origin. We derive expressions for the number of geodesics returning at t_n in two cases. First, as introductory example we will discuss this for $M = \mathbb{R}^2/\mathbb{Z}^2$ (a more complete discussion of this example can be found in [Pe3]). Second, we will deal with a much more unusual example, namely $M = \mathbb{H}^2/\Gamma(2)$, where $\Gamma(2)$ is a subgroup of $PSL(2, \mathbb{Z})$ called the principal congruence subgroup of level 2 (defined in more detail below). We note that it seems to be considerably harder to count geodesics that focus in points other than the origin.

Before continuing, consider the classical problem of counting $R_g(n)$, the number of solutions in $\mathbb{Z}^2$ of

$$p^2 + q^2 = n.$$

Let

$$n = 2^\alpha \prod_{i=1}^{k} p_i^{\beta_i} \prod_{j=1}^{\ell} q_i^{\gamma_i}$$

be the prime decomposition of the number n, where $p_i \equiv 1(\bmod 4)$ and $q_i \equiv 3(\bmod 4)$. The following classical result of Gauss (see [NZM]) will be very useful.

Lemma 4.1 *$R_g(n)$ is zero whenever n is not an integer, or any of the γ_i is odd. Otherwise,*

$$R_g(n) = 4 \prod_{i=1}^{k} (1 + \beta_i).$$

Example 4.2 Choose an origin in $M = \mathbb{R}^2/\mathbb{Z}^2$ and lift it to the origin in $\mathbb{R}^2$. We have that $S = \mathbb{Z}^2$. Let $\rho_x(t)$ be the number of geodesics of length t that connect the origin to the point $x \in M$.

Proposition 4.3 *In the flat torus, the number of geodesics of length t that connect the the origin to itself is given by*

$$\rho_0(t) = \begin{cases} R_g(t^2), & \text{if } t^2 \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Notice that by definition geodesics of length t leaving from the origin in $\mathbb{R}^2$ reach the points contained in $C_t(0)$. Only if t^2 is an integer does this circle intersect points in $\mathbb{Z}^2$. □

Example 4.4 We now turn to the next example. Recall that $PSL(2, \mathbb{Z})$ can be identified with the group of two by two matrices with integer entries and determinant one, and with multiplication by -1 as equivalence. The group $\Gamma(2)$ is an order 3 subgroup of $PSL(2,\mathbb{Z})$ obtained as the quotient of $PSL(2,\mathbb{Z})$ by the order 3 subgroup generated by $z \mapsto \frac{-1}{z-1}$.

$$\Gamma(2) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL(2,\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \,(\mathrm{mod}\ 2),\ b \equiv c \equiv 0 \,(\mathrm{mod}\ 2), \right\}.$$

This group has important applications in number theory. The action of $\Gamma(2)$ on $\mathbb{H}^2$ is given by the Möbius transformations

$$g(z) = \frac{az+b}{cz+d},$$

where $\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(2)$. We need a well-known result, which we reproduce here for completeness.

Lemma 4.5 *The orbit S of 0 in $\mathbb{D}^2$ under the fundamental group of $M = \mathbb{H}^2/\Gamma(2)$ is*

$$S = \left\{ \frac{p+iq}{r+is} \,\middle|\, p,q,r,s \in \mathbb{Z},\ p^2+q^2+1 = r^2+s^2,\ q+s \equiv 0 \,(\mathrm{mod}\ 2) \right\}.$$

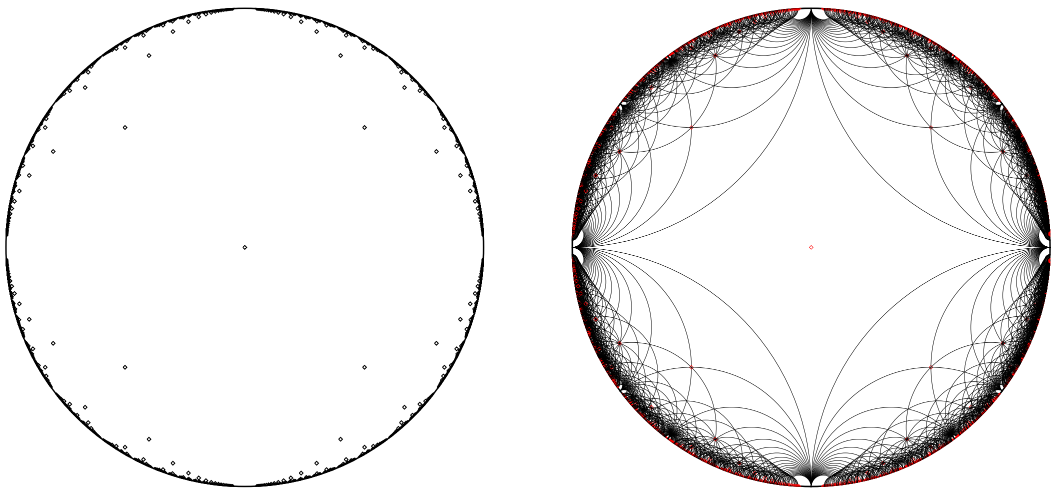

Figure 4.1: The orbit of 0 under $\Gamma(2)$ in the hyperbolic disk, and the corresponding Brillouin zones.

Proof: Following the conventions in [Be], define

$$\phi : \mathbb{D}^2 \to \mathbb{H}^2, \quad \phi(z) = i\frac{z+1}{-z+1}.$$

Push back the transformation $g \in \Gamma(2)$ from $\mathbb{H}^2$ to $\mathbb{D}^2$ by $g \to \phi^{-1} g \phi$ to obtain a representation of $g \in \Gamma(2)$ as a transformation in $SU(1,1)$ acting on $\mathbb{D}^2$. The matrix representation of this transformation is given by:

$$A_g = \left(\begin{array}{cc} \frac{a+d}{2} + i\frac{b-c}{2} & \frac{a-d}{2} - i\frac{b+c}{2} \\ \frac{a-d}{2} + i\frac{b+c}{2} & \frac{a+d}{2} - i\frac{b-c}{2} \end{array} \right),$$

where $\det A_g = 1$, since this matrix is conjugated to g, whose determinant is equal to 1. Let

$$p = (a-d)/2 \qquad q = -(b+c)/2$$
$$r = (a+d)/2 \qquad s = -(b-c)/2$$

and A_g now written as

$$A_g = \begin{pmatrix} r-is & p+iq \\ p-iq & r+is \end{pmatrix}.$$

Here the numbers p, q, r, s are in $\mathbb{Z}$ and must satisfy the following parity conditions:

$$\begin{aligned} p+r &\equiv 1 \pmod 2 \\ q+s &\equiv 0 \pmod 2. \end{aligned}$$

Recall that the determinant of A_g is equal to 1, therefore

$$p^2 + q^2 + 1 = r^2 + s^2.$$

It is easy to see that this last equation together with any one of the above parity conditions, say the one satisfied by p and r (q and s), imply the parity relation on q and s (p and r). This proves our claim. □

We need another auxiliary result before we state the main result of this section.

Lemma 4.6 *Let (p, q) and (r, s) be two points in $\mathbb{Z}^2$, such that the integers $A = p^2 + q^2$ and $B = r^2 + s^2$ are relatively prime, and let φ be a rotation. Now $\varphi(p, q) = (p', q')$ and $\varphi(r, s) = (r', s')$ are in $\mathbb{Z}^2$ if and only if φ is a rotation by a multiple of $\pi/2$.*

Proof: Let c be the cosine of the angle of rotation. We have

$$c = \frac{p'p + q'q}{A} = \frac{r'r + s's}{B}.$$

Thus if $p'p + q'q$ and $r'r + s's$ are not both equal to zero,

$$\frac{p'p + q'q}{r'r + s's} = \frac{A}{B}.$$

Because A and B are relatively prime and surely $|p'p + q'q|$ is less than or equal to A, and similarly for B, we have that

$$p'p + q'q = \pm A \quad \text{and} \quad r'r + s's = \pm B.$$

This implies the result. □

Now we define a counter just as before. Choose a lift of $M = \mathbb{D}^2/\Gamma(2)$ so that $0 \in M$ lifts to $0 \in \mathbb{D}^2$. Let $\gamma_x(t)$ be the number of geodesics of length t that connect the origin to the point $x \in M$.

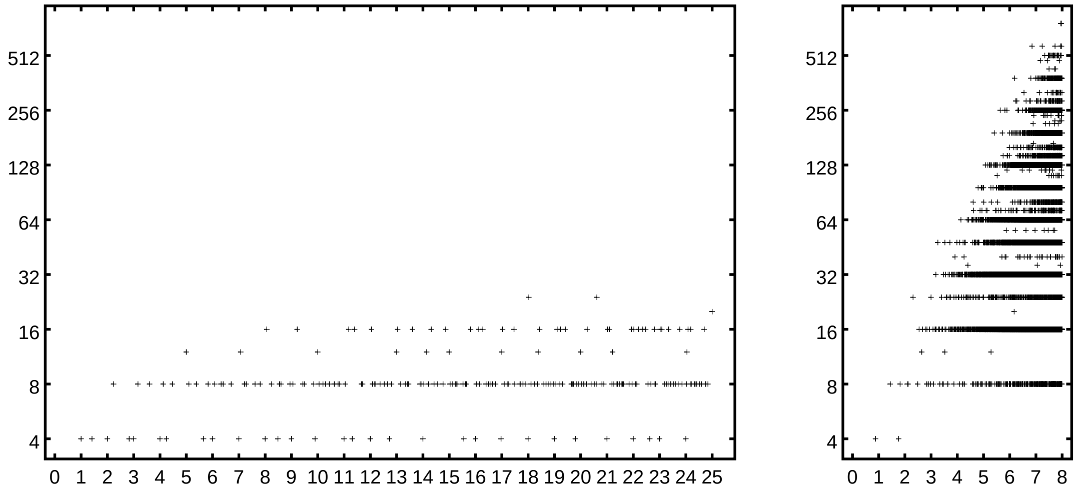

Figure 4.2: The non-zero values of $\rho_0(t)$ for $t \leq 25$ (left) and $\gamma_0(t)$ for $t \leq 8$ (right), which count how many geodesics of length t connect the origin to itself in the $\mathbb{R}^2/\mathbb{Z}^2$ and $\mathbb{D}^2/\Gamma(2)$, respectively.

Theorem 4.7 *For $\mathbb{D}^2/\Gamma(2)$, the number of geodesics of length t connecting the origin to itself is given by*

$$\gamma_0(t) = \begin{cases} \frac{1}{4}R_g(\cosh^2 t - 1)R_g(\cosh^2 t) & \text{if } \cosh^2 t \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We need to count how many distinct points of S (the orbit of $0 \in \mathbb{D}^2$) are contained in the circle $C_t(0)$ of radius t and centered at the origin.

A point $x = \frac{p+iq}{r+is}$ has Euclidean distance to the origin given by

$$|x|_e^2 = \frac{p^2 + q^2}{r^2 + s^2}.$$

Moreover, if $x \in S$, then by Lemma 4.5, we may define a positive integer n as follows:

$$n \equiv p^2 + q^2 = r^2 + s^2 - 1.$$

We will first count the number of 4-tuples (p, q, r, s) that solve this equation.

The hyperbolic length of the geodesic connecting the origin to this point is

$$t = \operatorname{arctanh}(|x|_e).$$

This gives that $\gamma_0(t)$ is only non-zero when $t = \operatorname{arctanh}\sqrt{\frac{n}{n+1}}$, or, equivalently, $n = \cosh^2 t - 1$. To count the number of intersections of $C_t(0)$ with S for these values of t, observe that we can use Gauss' result to count the number of pairs (p, q) such that $p^2 + q^2 = n$. This number is given by $R_g(n)$. For each such pair (p, q), we have a number of choices to form

$$x = \frac{p + iq}{r + is}$$

By the above, this number is equal to $R_g(n+1)$. Thus, $\gamma_0(t)$ is at most $R_g(n)R_g(n+1)$.

However, we have over-counted: some of our choices for p, q, r, s represent the same point $x \in S$, and some of them may not satisfy the parity condition $q \equiv s \pmod 2$. We will first account for the multiple representations, and then account for the parity.

Let $p, q, r, s \in \mathbb{Z}$ be as above, giving a point $x = \frac{p+iq}{r+is}$ at distance $t = \operatorname{arctanh}\sqrt{\frac{n}{n+1}}$ from the origin. If we multiply the numerator and denominator of x by $e^{i\theta}$, then x will remain unchanged. Because of the requirement that $p^2+q^2 = r^2+s^2-1 = n$, this is the only invariant, and by Lemma 4.6, θ must be a multiple of $\frac{\pi}{2}$ for the numerator and denominator to remain Gaussian integers. We see that in our counting, we have represented our point x in 4 different ways:

$$x = \frac{p+iq}{r+is} = \frac{-q+ip}{-s+ir} = \frac{-p-iq}{-r-is} = \frac{q-ip}{s-ir}$$

meaning we have over-counted by a factor of at least 4.

Finally, note that if the representation $\frac{p+iq}{r+is}$ fails to satisfy our parity condition, then $q+s \equiv 1 \pmod 2$ and consequently $p+r \equiv 0 \pmod 2$. This means that the representation $\frac{-q+ip}{-s+ir}$ of this same point does satisfy the parity conditions, giving exactly $\frac{1}{4}R_g(\cosh^2 t - 1)R_g(\cosh^2 t)$ distinct points of S at distance t from the origin. □

Minor modifications to this argument give exact counts on the number of geodesics of length t which connect the origin to itself for $\mathbb{D}^2/\Gamma(3)$ and $\mathbb{D}^2/\Gamma(5)$. We shall do this now. Let

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \pmod N,\ b \equiv c \equiv 0 \pmod N \right\}.$$

Transporting this group to $\mathbb{D}^2$ as before, we obtain the representation

$$\left\{ \begin{pmatrix} r-is & p+iq \\ p-iq & r+is \end{pmatrix} \,\middle|\, \begin{array}{ll} r+p \equiv 1 \pmod N, & r-p \equiv 1 \pmod N \\ s+q \equiv 0 \pmod N, & s-q \equiv 0 \pmod N \end{array} \text{ and } p^2+q^2+1 = r^2+s^2 \right\}.$$

Theorem 4.8 *The number of geodesics of length t which connect the origin to itself in the surfaces $\mathbb{D}^2/\Gamma(3)$ and $\mathbb{D}^2/\Gamma(5)$ are given by*

$$\frac{1}{4}R_g\left(\frac{\cosh^2 t - 1}{9}\right)R_g(\cosh^2 t) \qquad \textit{and} \qquad \frac{1}{4}R_g\left(\frac{\cosh^2 t - 1}{25}\right)R_g(\cosh^2 t)$$

respectively.

Proof: First, let us consider $N = 3$. Since N is odd, the congruence conditions on r, p, s, and q imply that

$$r \equiv 1 (\text{mod } 3) \qquad \text{and} \qquad p \equiv q \equiv s \equiv 0 (\text{mod } 3).$$

Note that the equation

$$p^2+q^2 = n \qquad \text{and} \qquad p \equiv q \equiv 0 \pmod 3$$

is satisfied exactly $R_g(n/3^2)$ times. (Recall that if $n/9$ is not an integer, $R_g(n/9)=0$.) For fixed n, let (p,q) be any one of the solutions. We need to decide how many solutions the equation

$$r^2+s^2=n+1 \quad \text{with} \quad r\equiv 1\,(\mathrm{mod}\ 3) \quad \text{and} \quad s\equiv 0\,(\mathrm{mod}\ 3)$$

admits. The solution of the first equation implies that 3 divides n. Thus $r^2+s^2\equiv 1\,(\mathrm{mod}\ 3)$. Consequently, there are 4 choices mod 3 for the pair (r,s), namely $(0,1)$, $(1,0)$, $(0,2)$, and $(2,0)$.

Let $(p,q,r,s)\in\mathbb{Z}^2\times\mathbb{Z}^2$ be any solution to $n=p^2+q^2=r^2+s^2-1$ with $p\equiv q\equiv 0(\mod 3)$. For each choice of (p,q), we have exactly $R_g(n+1)$ choices of (r,s). Now let R denote the product of the rotations by $\pi/2$ on each of the components of $\mathbb{Z}^2\times\mathbb{Z}^2$. Using Lemma 4.6, we see that all such solutions can be obtained from just one by applying R repeatedly. It is easy to check that each quadruple of solutions thus constructed runs exactly once through the above list. Since precisely one out of the four associated solutions is compatible with the conditions, the total number of solutions is exactly:

$$\frac{1}{4}R_g\left(\frac{n}{9}\right)R_g(n+1).$$

Using the relationship between the Euclidean distance and the Poincaré length as before gives the result.

For $N=5$, the proof can be literally transcribed to obtain the second result. □

Remark: Note that the above results do not hold if N is not one of the cases mentioned. The primary difficulty is that for prime $N\geq 7$, there are solutions which are not related by applying the rotation R. However, the argument does give $\frac{1}{4}R_g\left((\cosh^2 t-1)/N^2\right)R_g(\cosh^2 t)$ as an upper bound for $\mathbb{D}^2/\Gamma(N)$ when $N\geq 7$ is prime.

5 Non-Riemannian examples

The present context is certainly not restricted to Riemannian metrics. As an indicator of this we now discuss a different set of examples.

Let k be a positive number greater than one. Equip $\mathbb{R}^2$ with the distance function

$$\| x-y \|=\left(|x_1-y_1|^k+|x_2-y_2|^k\right)^{1/k}$$

and let the discrete set S be given by $\mathbb{Z}^2$. For k not equal to 2, this is not a Riemannian metric, yet all conclusions of section 2 hold. In particular, each Brillouin zone forms a fundamental domain. Note that determining the zones by inspecting the picture requires close attention!

Now the problem of determining $C_t(0)\cap S$ for any given t is unsolved for general k. In fact, even for certain integer values of k greater than 2, it is not known whether $C_t(0)\cap S$ ever contains at least two points that are not related by the symmetries of the problem. For $k=4$, the smallest t for which $C_t(0)\cap S$ has at least two (unrelated) solutions is given by

$$t^4=133^4+134^4=158^4+59^4.$$

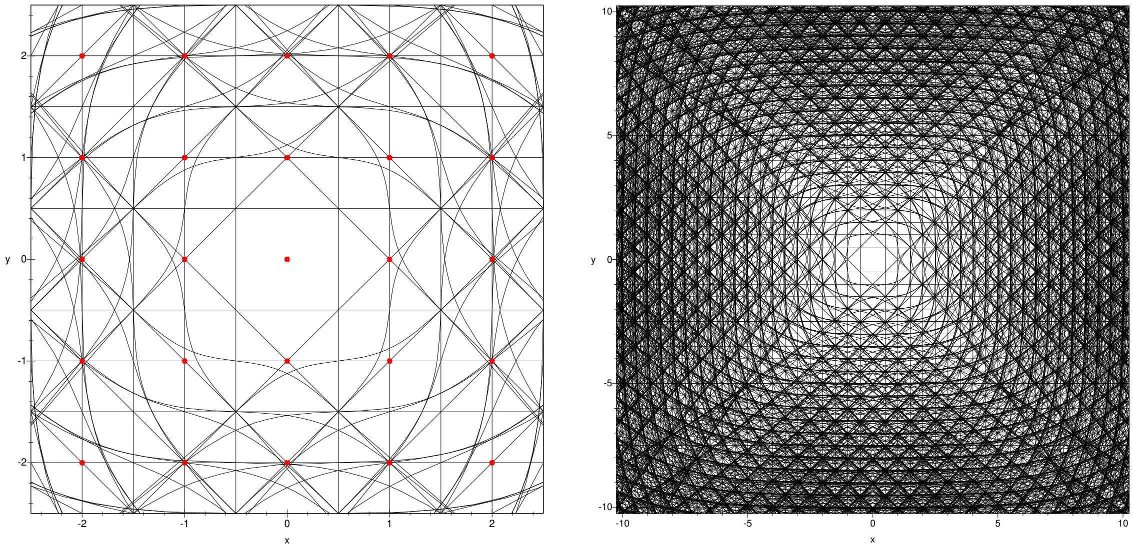

Figure 5.1: Brillouin zones for the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$ with the metric $\left(|x_1 - y_1|^4 + |x_2 - y_2|^4\right)^{1/4}$. See also Figure 2.3 and Example 2.6, which deal with the case $k = 1$, the "Manhattan metric".

However, for $k \geq 5$, it unknown whether this can happen at all (see [SW]).

There are some things that can be said, however. In the situation where $k \in \{3, 4, 5, \ldots\}$, the mediatrices intersect the coordinate axes only in irrational points or in multiples of $1/2$. For if $x = (p/q, 0)$ is a point of a mediatrix $L_{(0,0),(a_1,a_2)}$, we have

$$|p|^k = |p - qa_1|^k + |a_2 q|^k \quad (p \neq 0, q \neq 0) \quad .$$

By Fermat's Last Theorem, this has no solution unless either $p = qa_1$ or $a_2 = 0$. In the first case, $\frac{p}{q} = \pm a_2$, which can only occur if the lattice point is of the form $(a_2, \pm a_2)$. If $a_2 = 0$, then $\frac{p}{q} = \frac{a_1}{2}$. In particular, there is no nontrivial focusing along the axes.

To compute Figure 5.1, we took advantage of the smoothness of the metric. Not all metrics are sufficiently smooth for this procedure to work. Even for Riemannian metrics, in general the distance function is only Lipschitz, which will not be sufficiently smooth.

For each $a = (a_1, a_2) \in \mathbb{Z}^2$, define a Hamiltonian:

$$H_a(x) = \parallel x - a \parallel - \parallel x \parallel .$$

The mediatrix L_{0a} corresponds to the level set $H_a(x) = 0$. Because $H_a(x)$ is smooth, we have uniqueness of solutions to Hamilton's equations. In the current situation, where the dimension is two, the level set consists of one orbit. Thus, one can produce the mediatrix by numerically tracing the zero energy orbits of the above Hamiltonian.

As mentioned above, for a general Riemannian metric, the distance function is only Lipschitz. This means we have no guarantee that the solutions of the above differential equation are unique. Indeed, there are examples of multiply connected Riemannian manifolds with self-intersecting mediatrices, as will be shown in a forthcoming work.

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