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DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS ON METRIC MEASURE SPACES

J. Cheeger

Contents

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0 Introduction

In this paper, we extend to certain metric measure spaces, much of that part of calculus which concerns first derivatives. In particular, we give a generalization of the theorem of Rademacher which asserts that a real valued Lipschitz function on $\mathbf{R}^n$ is differentiable almost everywhere with respect to Lebesgue measure; see [R]. Thus, we show that in a suitably generalized sense, at almost all points, the blow ups of a real valued Lipschitz function converge to a unique linear function. As a consequence, it follows that the underlying space possesses a degree of small scale and infinitesimal regularity. There are many natural examples of spaces to which our discussion applies, including fractal spaces of Hausdorff dimension, d, for every real number, $1 < d < \infty$; see [Bo], [BoP], [Gro2], [L].

When specialized to $\mathbf{R}^n$, our generalized notion of differentiability coincides with the usual one; see Theorems 4.38, 8.11, 10.2. In particular, our results provide a proof of the classical Rademacher theorem, which although far from being the most direct, is genuinely new; see Remark 16.36 for further discussion. More significantly, we give an *explanation* for the validity of Rademacher's theorem, on the basis of simple general conditions.

Our most basic assumptions are that Z is a metric space and that μ is a Borel regular measure on Z. From now on, we consider only Borel regular measures which are finite and nonzero on balls of finite nonzero radius.

It is perhaps surprising that there exists a *partial* generalization of Rademacher's theorem in which the only assumption is that the measure, μ, satisfies the Vitali covering theorem; see Theorem 3.7. However, for such spaces, there might not exist any rectifiable curves. If there do not exist "sufficiently many" such curves (as measured with respect to μ) the conclusion of our partial generalization holds for trivial reasons at *all* points, and hence, does not give any nontrivial constraint. The Vitali covering theorem holds for *any* Radon measure on $\mathbf{R}^n$ and even for some Gaussian measures on Hilbert space; see Chapter 2 of [Ma] and the references therein.

Recall in particular, that the Vitali covering theorem is implied by the doubling condition on the measure μ. The doubling condition holds it for all $0 < r'$, there exists $\kappa = \kappa(r')$, such that for all $z \in Z$ and $0 < r < r'$,

$$\mu(B_r(z)) \leq 2^{\kappa} \mu(B_{r/2}(z)) \,. \tag{0.1}$$

(If κ can be chosen independent of r we would say that a *global* doubling condition holds, but our present considerations are essentially local in nature.)

Given (0.1) and the existence of a weak Poincaré inequality of type $(1, p)$, for some $1 \le p < \infty$, we prove two results which when taken together and specialized to Euclidean space, immediately imply the classical theorem of Rademacher; see Theorems 4.38, 10.2. The Poincaré inequality (see (4.3)) is formulated in terms of the notion of "upper gradient", introduced in [HeKo2] and reviewed in section 1.

A geometric condition which is sufficient (and close to being necessary) for a Poincaré inequality to hold is (roughly speaking) that there exists a "thick" family of "not too long" paths between each pair of points; see [Se3]. This condition will not play a direct role in the approach adopted here; compare however sections 15, 16, where a refined version of this condition is assumed in order to obtain certain quantitative results; see Definition 15.15.

Recall that Z is called λ-quasi-convex if for all $z_1, z_2 \in Z$, there exists a rectifiable curve from z_1 to z_2 of length at most the $\lambda\overline{z_1, z_2}$, where $\overline{z_1, z_2}$ denotes the distance from z_1 to z_2. Canonically associated to a λ-quasi-convex metric, is a length space metric, which is λ-bi-Lipschitz to the original one.

According to an observation of David and Semmes, if Z is complete and (Z, μ) satisfies a doubling condition and weak Poincaré inequality, then every ball, $B_r(\underline{z})$, with $0 < r < r'$, is $c(\kappa, C)$-quasi-convex, with $\kappa = \kappa(r')$ as in (0.1) and $C = C(r')$, the constant in the weak Poincaré inequality; see (4.3). Thus, if the doubling condition and Poincaré inequality hold, then for considerations which are invariant under $c(\kappa, C)$-bi-Lipschitz equivalence, it suffices to assume that Z is a length space; compare also the proof of Theorem 6.1.

For completeness, a proof of the result of David-Semmes (essentially the one which was communicated to the author by Stephen Semmes) will be given in the Appendix; compare also the discussion just after Proposition 6.12 of [DSe2] and see Lemma 2.38 of [Se5] for Semmes' exposition.

Examples of Ahlfors regular (and in particular, doubling) *fractal* length spaces, of integral Hausdorff dimension for which the Poincaré inequality holds, are provided by the Carnot-Caratheodory spaces of [Gro2]. As shown in [BoP], corresponding examples of nonintegral Hausdorff dimension are given by the boundaries of certain 2-dimensional hyperbolic buildings. In [L], T.J. Laakso constructs (by hand) spaces with these same properties in every Hausdorff dimension d, for $1 < d < \infty$. His spaces are obtained as quotients by finite to one maps, of products of intervals with various Cantor sets. None of the above mentioned fractal spaces admits a bi-Lipschitz imbedding into $\mathbf{R}^N$, for any N; see section 14 and [Se4].

For $r > 0$, put $f_{r,\underline{x}} = (f - f(\underline{x}))/r$. To say that a function, $f : \mathbf{R}^n \to \mathbf{R}$, is differentiable at $\underline{x} \in \mathbf{R}^n$ means the following. If we rescale the distance on $\mathbf{R}^n$, replacing d by $r^{-1}d$, then as $r \to 0$, the functions, $f_{r,\underline{x}}$, converge uniformly on compact subsets, to a unique limit function, $f_{0,\underline{x}}$, on the tangent space at $\underline{x}$. Moreover, the limit function, $f_{0,\underline{x}}$ is linear.

Since in Euclidean space, any two balls with the same center can be identified via radial projection, one can define *unambiguously* the sense in which the limit function is unique.

Let x_i denote the i-th coordinate function. A second way of expressing the uniqueness condition in the definition of differentiability is to say that there exist constants $b_1, \ldots, b_n$, such that

$$\limsup_{r\to 0} \sup_{B_r(\underline{x})} \left| - f_{r,\underline{x}} + b_1(x_1)_{r,\underline{x}} + \cdots + b_n(x_n)_{r,\underline{x}} \right| = 0 \tag{0.2}$$

Indeed, if (0.2) holds and for some $a_1, \ldots, a_n$, we have

$$\liminf_{r\to 0} \sup_{\overline{x,\underline{x}}=r} \left| - f_{r,\underline{x}} + a_1(x_1)_{r,\underline{x}} + \cdots + a_n(x_n)_{r,\underline{x}} \right| = 0\,, \tag{0.3}$$

then

$$\liminf_{r\to 0} \sup_{\overline{x,\underline{x}}=r} \left| (b_1 - a_1)(x_1)_{r,\underline{x}} + \cdots + (b_n - a_n)(x_n)_{r,\underline{x}} \right| = 0\,, \tag{0.4}$$

which implies, $a_i = b_i$, for all i.

Note that strictly speaking, the second formulation of uniqueness is a *relative* one; to recover first, we must use the fact that convergence to *unique* limit function, $(x_i)_{0,\underline{x}}$, has an unambiguous meaning for the functions, $(x_i)_{r,\underline{x}}$, and does hold for these functions.

If we take the view point that *by definition*, linear limit functions are precisely those which are linear combinations of limit functions, $(x_i)_{0,\underline{x}}$, then (0.2), in addition to implying the relative uniqueness of the limit function, implies the linearity of $f_{0,\underline{x}}$ as well. More generally, we would say that $f_{r,\underline{x}}$ is *asymptotically linear* (or f is asymptotically linear at $\underline{x}$) if (0.2) holds, with the constants, b_i, replaced by bounded functions, $b_i(r)$. However, this virtually tautologous characterization of asymptotic linearity suggests no immediate generalization. Thus, we will require a characterization which is more intrinsic in nature.

For Z a metric space, $\underline{z} \in Z$, the functions, $f_{r,\underline{z}}$, are defined as above. Our partial generalization of Rademacher's theorem asserts that if Z is a metric space and μ satisfies the Vitali covering theorem, then for any Lipschitz function, f, and μ-a.e. $\underline{z} \in Z$, the function, $f_{r,\underline{z}}$, is *asymptotically generalized linear* (which concept is expained below); see Theorem 3.7.

Under the additional assumptions that μ is doubling and a Poincaré inequality holds, we prove an assertion which corresponds to (0.2)–(0.4).

This entails the μ-a.e. (relative) uniqueness of limit functions, as well as the finite dimensionality of the space of such functions; see Theorem 4.38.

If the measure, μ, is doubling, then tangent cones, $Z_{\underline{z}}$, exist for all $\underline{z}$, but need not be unique. For f Lipschitz, limit functions, $f_{0,\underline{z}} : Z_{\underline{z}} \to \mathbf{R}$, exist for any $Z_{\underline{z}}$, provided we pass to suitable subsequences, $r_j \to 0$. By passing to subsequences, we can also equip any $Z_{\underline{z}}$ with a renormalized limit measure; compare [Fu], [ChCo2].

We show that if the doubling condition and Poincaré inequality hold, then for μ-a.e. $\underline{z}$, all limit functions, $f_{0,\underline{z}}$, are themselves *generalized linear*. Moreover, the pointwise Lipschitz constant $\operatorname{Lip} f_{0,\underline{z}}$, of any such limit function, $f_{0,\underline{z}}$, is a constant function, $\operatorname{Lip} f_{0,\underline{z}} \equiv \operatorname{Lip} f(\underline{z})$; see Theorem 10.2 and section 1 for the definition of the pointwise Lipschitz constant.

In our general context, the notion of linear function is not defined. To circumvent this difficulty, we first define a notion of *minimal generalized upper gradient*, which plays the role of the *norm* of the gradient (or differential); see the end of this introduction and section 2 for the definition. Then, we define *generalized linear function* to be either the function, $\ell \equiv 0$, or a Lipschitz function, ℓ, with range, $(-\infty, \infty)$, which is *harmonic* in the variational sense (of Dirichlet's principle) and for which the *minimal generalized upper gradient*, g_ℓ, satisfies $g_\ell \equiv c$, for some constant c.

Given a Lipschitz function, f, the pointwise Lipschitz constant, $\operatorname{Lip} f$, is always an upper gradient, and hence, a generalized upper gradient of f.

For Lipschitz functions on Euclidean space, the minimal generalized upper gradient, g_f of f, is indeed $\operatorname{Lip} f$. Moreover, as explained at greater length below, a generalized linear function on Euclidean space is a linear function in the usual sense.

In the presence of the doubling condition and the Poincaré inequality, we show initially that the pointwise Lipschitz constant, $\operatorname{Lip} f$, is bounded above μ-a.e. by a definite multiple of the minimal upper gradient. In Theorem 5.1, we show that if in addition, Z is a length space, then $g_f(\underline{z}) = \operatorname{Lip} f(\underline{z})$, for μ-a.e. $\underline{z}$. Finally, in Theorem 6.1, we remove the length space assumption. As a consequence of Theorem 6.1, for μ-a.e. $\underline{z}$, we have $\operatorname{Lip} f(\underline{z}) = \lim_{r\to 0} \sup_{\overline{z,\underline{z}}=r} |f(z) - f(\underline{z})|/r$; compare Proposition 1.11 and see Corollary 6.36. In particular, the limit on the right-hand side exists.

The role of the (intermediate) assumption that Z is a length space can be explained as follows: If Z is a length space (no measure specified) and g is a *continuous* upper gradient for f, then $\operatorname{Lip} f \leq g$. Indeed, given $\underline{z}$ and $\epsilon > 0$, there exists $\delta > 0$, such that $r < \delta$ implies $g_f < g_f(\underline{z}) + \epsilon$, on $B_r(\underline{z})$. If $\overline{z,\underline{z}} < \delta$, then by integrating the function, g_f, along a minimal geodesic (parameterized by arclength) from $\underline{z}$ to z, we get $|f(z) - f(\underline{z})| < (g_f(\underline{z}) + \epsilon)\overline{z,\underline{z}}$; see Definition 1.1. This gives our assertion.

In the length space case, the essential point is to deal with the possiblity that g might not be continuous and need only be a *generalized* upper gradient. For this we use a (nontrivial) approximation argument. Removing the length space assumption involves a similar approximation argument. In both instances, Mazur's lemma plays an essential role; see the discussion of the reflexivity of $H_{1,p}$ below.

Given the doubling condition on μ, it follows that a function, f, satisfies a reverse Poincaré inequality on sufficiently small balls centered at a point $\underline{z}$, at which f is asymptotically generalized linear and for which $g_f(\underline{z}) > 0$; see Theorem 3.14.

A related argument shows that on any domain, a generalized linear function can be recovered by a canonical procedure from its boundary values; see Theorem 8.5. This implies in particular, that every point lies on a geodesic line, γ, with $\gamma(0) = \underline{z}$, such that $\ell(\gamma(s)) = \operatorname{Lip}\ell \cdot s$, where s denotes arclength and, since ℓ is generalized linear, $\operatorname{Lip}\ell$ is a constant function. Here (as usual) a *line* is a doubly infinite geodesic, each finite segment of which is minimal. Thus, γ would be an integral curve of, $\nabla\ell$, the gradient of ℓ, if $\nabla\ell$ were actually defined.

Similarly, we find that $\ell(\underline{z}) - \operatorname{Lip}\ell \cdot b_{\underline{\gamma}} \leq \ell \leq \ell(\underline{z}) + \operatorname{Lip}\ell \cdot b_{-\underline{\gamma}}$, where $b_{\underline{\gamma}}, b_{-\underline{\gamma}}$, denote the Busemann functions associated with $\gamma, -\gamma$, respectively. From this it follows that on Euclidean space, generalized linear functions are precisely those which are linear in the usual sense. (More generally, for riemannian manifolds with $\operatorname{Ric}_{M^n} \geq 0$, they are coordinate functions corresponding to lines which split off as isometric factors; compare [ChGr].) We point out that the above mentioned argument does *not* require our assuming a priori, that generalized linear functions on Euclidean space are differentiable, e.g. because they are Lipschitz.

It follows from the reverse Poincaré inequality, the doubling condition on μ and the Poincaré inequality, that there is a definite upper bound for the dimension of any space of functions, which are almost generalized linear. This, together with the fact that $\operatorname{Lip} f$ is bounded μ-a.e. by a definite multiple of g_f, enables to show that that there exists a *finite dimensional cotangent bundle*, T^*Z, certain sections of which are the (suitably defined) *differentials*, df, of Lipschitz functions f; compare [W]. Typically, the transition functions of our cotangent bundle are only L_∞ and not continuous.

If the measure, μ, is Ahlfors regular, then μ-a.e., the Hausdorff dimension of Z is greater than or equal to the dimension of T^*Z. Conjecturally, in the case of equality, the space, Z, is *μ-rectifiable*; see Conjectures 4.63, 4.65 and compare the results of sections 12–14.

For a (suitable) Lipschitz map between two spaces satisfying our assumptions, there is an induced map on cotangent bundles. Under certain additional assumptions, the same holds for quasiconformal homeomorphisms.

One can define a norm on the sections of T^*Z such that $|df|_{L_p} = |g_f|_{L_p} = |\mathrm{Lip}\, f|_{L_p}$, for Lipschitz functions f. Typically, this norm does not arise from an inner product. This Finslerian (as opposed to riemannian) character of our spaces is, of course, inevitable, since for example, any finite dimensional normed linear space, equipped with Lebesgue measure, satisfies our assumptions.

The existence of the *finite dimensional* generalized cotangent bundle (and the concommitant uniqueness of strong derivatives) has strong consequences. In particular, it implies that suitably defined Sobolev spaces, $H_{1,p}$, are *reflexive*; see section 2 for the definition of $H_{1,p}$. This fact, which enables us to implement Mazur's lemma, plays an important role in proofs of Theorems 5.1, 6.1, 13.4; see also Theorem 4.53. As another consequence, it follows that the theory developed in [HeKM] for measures on Euclidean space which are given by what they call p-admissible weights, is applicable in our situation. (While this has been checked at considerable length, we do not give any details here.)

We show in section 9, that measured Gromov Hausdorff limits of sequences of spaces which satisfy a uniform doubling condition and Poincaré inequality, also satisfy these conditions with the same constants. In particular, this holds for tangent cones. It follows that the dimension of the space of generalized linear functions on any iterated tangent cone has an a priori upper bound, N. Thus, for any integer, m, it follows that any sequence of iterated tangent cones, $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{(m-1)N+m}$, contains an iterated tangent cone, $\mathcal{C}_j$, such that the maximal dimension of a space of generalized linear functions on $\mathcal{C}_j$ is equal to the maximal dimension of such a space on any tangent cone at a point of $\mathcal{C}_{j+m}$. This property of $\mathcal{C}_j$ is shared by Minkowski spaces, i.e. finite dimensional normed linear spaces. In this way, our discussion yields restrictions on the infinitesimal geometric structure our space, Z, at μ-a.e. $\underline{z} \in Z$.

In section 12, we give sufficent condition for a norm on T^*Z to be given by the pointwise Lipschitz constant of some length space metric.

We call the space, $(T^*Z_{\underline{z}})^*$, the *tangent space* at $\underline{z}$, and denote this space by $TZ_{\underline{z}}$. In section 13, using the results of section 12, we show that for μ-a.e. $\underline{z} \in Z$ and any tangent cone, $Z_{\underline{z}}$, there exists an essentially canonical *surjective* Lipschitz map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$. It follows that the Hausdorff dimension of any such tangent cone, $Z_{\underline{z}}$, is no smaller than the dimension of TZ_z.

The result of section 13 plays a role in the context of the bi-Lipschitz nonimbedding theorem of section 14. There, it is shown in particular, that if such an imbedding exists, then for μ-a.e. $\underline{z} \in Z$, the map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$, is a bi-Lipschitz equivalence.

Sections 15, 16 are concerned with a class of length spaces which satisfy what we call (ϵ, δ)-inequalities, for all ϵ, δ. We show in section 15 that for such spaces it follows — much more easily than in section 5 and in *quantitative* form — that $\operatorname{Lip} f$ is the minimal generalized upper gradient of f.

Important examples of spaces which satisfy an (ϵ, δ)-inequality for all ϵ, δ, are those which we call *thickly minimally connected.* Thickly minimally connected length spaces are also easily seen to satisfy the doubling condition, (0.1). All of examples to which we alluded earlier are actually thickly minimally connected. Moreover, from the directionally restricted relative volume comparison property, one also sees that smooth manifolds satisfying $\mathrm{Ric}_{M^n} \geq -(n-1)$, including of course, $\mathbf{R}^n$, are (uniformly) thickly minimally connected.

The results of section 15 are used in section 16. In particular, we give a quantitative analog of Theorem 8.5, which concerns the canonical representation of a generalized linear function in terms of its boundary values; see 16.32. The results of section 16 have significant applications in the context of [ChCoMi] and [ChCo3].

We close this introduction by briefly describing the proof of the *partial* generalization of Rademacher's theorem which requires only that the Vitali covering theorem hold for the measure, μ. As indicated earlier, absent the existence of sufficiently many rectifiable curves, the function, g_f, defined below might vanish identically for functions, f, which are not locally constant.

Let $U \subset Z$ be open and $f : U \to \mathbf{R}$. For $p \geq 1$, put $|f|_{1,p} = \inf_{g_i} |f|_{L_p} + |g_i|_{L_p}$, where the infimum is taken over all sequences, $f_i \xrightarrow{L_p} f$, and all upper gradients, g_i, of f_i. This definition makes Rellich's theorem a virtual tautology:

If $f_i \xrightarrow{L_p} f$, then $\liminf |f_i|_{1,p} \geq |f|_{1,p}$. One also easily checks that the $(1,p)$-norm is nonincreasing under truncation, i.e. $|\min(f,c)|_{1,p} \leq |f|_{1,p}$, for any constant, c.

For $1 < p < \infty$, it follows from the uniform convexity of L_p, that there exists a minimal generalized upper gradient, $g_f : U \to [0,\infty]$, unique up to modification on certain subsets of measure zero, such that $|f|_{1,p} = |f|_{L_p} + |g_f|_{L_p}$. Moreover, one verifies that for all $A \subset U$, if $f_1 \,|\, A = f_2 \,|\, A$, then $g_{f_1}(z) = g_{f_2}(z)$, for μ-a.e. $z \in A$.

Now suppose that the Vitali covering theorem holds. If for f Lipschitz, we assume that $f_{r,\underline{z}}$ does not satisfy the condition of harmonicity, asymptotically as $r \to 0$, for μ-a.e. $\underline{z}$, then, by means of a replacement argument, together the fact that the $(1,p)$-norm does not increase under truncation, Rellich's theorem can easily be contradicted. Since at Lebesgue points of g_f, this function approaches a limit in the sense of Lebesgue's theorem, we conclude that the functions, $f_{r,\underline{z}}$, are asymptotically generalized linear, for μ-a.e. $\underline{z}$.

For previous works which are concerned with the extension of methods of analysis on Euclidean spaces to the context of metric measure spaces, we refer, for example, to [CoiWe1,2], [GroLaPa], [DSe1,2], [F], [H], [HKo], [HeKM], [HeKo1,2], [KiM], [Se1–5], [St].

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1 Upper Gradients and Lipschitz Functions

Let Z be a metric space. We begin by recalling the concept of upper gradient introduced in [HeKo2].

Fix a set, $A \subset Z$. Let f be a function on A with values in the extended real numbers.

Definition 1.1 ([HeKo2]). An *upper gradient*, g, for f is an extended real valued Borel function, $g : A \to [0,\infty]$, such that for all points, $z_1, z_2 \in A$ and all continuous rectifiable curves, $c : [0,\ell] \to A$, parameterized by arclength, s, with $c(0) = z_1$, $c(\ell) = z_2$, we have

$$\big|f(z_2) - f(z_1)\big| \leq \int_0^\ell g(c(s))ds\,. \tag{1.2}$$

Note that in Definition 1.1, the left-hand side is interpreted as ∞, if either $f(z_1) = \pm\infty$ or $f(z_2) = \pm\infty$; compare the discussion following (5.8). If on the other hand, the right-hand side of (1.2) is finite, then it follows that $f(c(s))$ is a continuous function of s. In the present paper, we will only be concerned with the case in which f is a Borel function. Typically, A will be an open subset (often denoted by U).

REMARK 1.3. If, as holds for discrete spaces or "snow flakes", Z contains no nonconstant rectifiable curves, then for any f, we can take $g \equiv 0$. In such cases, g does not exert any control over the behavior of f.

REMARK 1.4. If Z is rectifiably connected, then there is a length space metric, on Z, canonically associated to the given metric. It is easy to verify that a curve, c, is rectifiable of length, ℓ, with respect to one of these metrics, if and only if the same holds for the other. Hence, a function, g, is an upper gradient for a function, f, with respect to one of these metrics, if and only if the same holds for the other.

The following two propositions are easily verified.

PROPOSITION 1.5. *If g_1, g_2 are upper gradients for f_1, f_2, then $|\alpha_1|g_1 + |\alpha_2|g_2$ is an upper gradient for $\alpha_1 f_1 + \alpha_2 f_2$. Moreover, $\max(g_1, g_2)$ is an upper gradient for $\max(f_1, f_2)$ and for $\min(f_1, f_2)$.*

PROPOSITION 1.6. *Let $V_1, V_2 \subset U$ be open and let $g_j : V_j \to [0,\infty]$ be an upper gradient for the restriction of $f : V_1 \cup V_2 :\to \mathbf{R}$ to V_j. Extend g_j to $V_1 \cup V_2$ by setting $g_j \,|\, (V_1 \cup V_2) \setminus V_j \equiv 0$. Then the function, $\max(g_1, g_2)$ is an upper gradient for f.*

For completeness, we indicate the elementary proof of the following basic lemma.

LEMMA 1.7. *Let g_1, g_2 be upper gradients for f_1, f_2 respectively. Then for all $\epsilon > 0$, the function, $g_1(|f_2| + \epsilon) + (|f_1| + \epsilon)g_2$, is an upper gradient for $f_1 f_2$.*

Proof. Let c be a rectifiable curve, $c : [0,\ell] \to U$. We can assume that the restrictions of both g_1 and g_2 to the curve, c, are integrable, and hence, that the restrictions of f_1, f_2 to this curve are uniformly continuous. Otherwise, there is nothing to prove. Fix $n > 0$ and put $\ell_i = (i\ell)/n$, where $i = 0, \ldots, n-1$. We have

$$\big|f_1(c(\ell_{i+1}))\big|\,\big|f_2(c(\ell_{i+1})) - f_2(c(\ell_i))\big| \le \big|f_1(c(\ell_{i+1}))\big| \int_{\ell_i}^{\ell_{i+1}} g_2(c(s))ds\,, \tag{1.8}$$

$$\big|f_1(c(\ell_{i+1})) - f_1(c(\ell_i))\big|\,\big|f_2(c(\ell_i))\big| \le \big|f_2(c(\ell_i))\big| \int_{\ell_i}^{\ell_{i+1}} g_1(c(s))ds\,. \tag{1.9}$$

If we add these equations, sum over i and use the uniform continuity of f_1, f_2, the lemma easily follows. □

Define the extended real valued Borel function, $Lip\, f$, by

$$Lip\, f(\underline{z}) = \liminf_{r\to 0} \sup_{\overline{z,\underline{z}}=r} \frac{|f(z) - f(\underline{z})|}{r}\,, \tag{1.10}$$

where we put $Lip\, f(\underline{z}) = 0$ if $\underline{z}$ is isolated. For f Lipschitz, the function, $Lip\, f$, is finite and bounded above by the Lipschitz constant, $\mathbf{Lip}\, f$, of f.

The following proposition, which is a restatement of Lemma 1.20 of [Se1], provides an important example of an upper gradient, for the case in which f is Lipschitz.

PROPOSITION 1.11. *If f is Lipschitz then $Lip\, f$ is an upper gradient for f.*

Proof. Since the restriction of f to any rectifiable curve, c, is Lipschitz and in particular, absolutely continuous, it follows that $f(c(s)) = f_c(s)$ is differentiable for almost every s. Moreover, (1.2) holds, with $g(c(s))$ replaced by $|f_c'(s)|$. Thus, it suffices to show $|f_c'(s)| \le Lip\, f(c(s))$, for those s, for which $f_c'(s)$ exists.

Fix such a value $\underline{s} < \ell$. We can assume that $\overline{c(s), c(\underline{s})} \not\equiv 0$, for s sufficiently close to $\underline{s}$, since otherwise, $f_c'(\underline{s}) = 0$. Then, by continuity, for all sufficiently small r, there exists some smallest $s(r)$, with $\overline{c(s(r)), c(\underline{s})} = r \le |s(r) - \underline{s}|$. In addition, $s(r) \to 0$ as $r \to 0$. We have

$$\frac{|f_c(s(r)) - f_c(\underline{s})|}{|s(r) - \underline{s}|} \le \sup_{\overline{z, c(\underline{s})}=r} \frac{|f(z) - f(c(\underline{s}))|}{r}\,. \tag{1.12}$$

Since $f_c'(\underline{s})$ exists, we get

$$\begin{aligned} |f_c'(\underline{s})| &= \liminf_{r\to 0} \frac{|f_c(s(r)) - f_c(\underline{s})|}{|s(r) - \underline{s}|} \\ &\le \liminf_{r\to 0} \sup_{\overline{z, c(\underline{s})}=r} \frac{|f(z) - f(c(\underline{s}))|}{r} \\ &= Lip\, f(c(\underline{s}))\,, \end{aligned} \tag{1.13}$$

and our assertion follows. □

We also put

$$lip\, f(\underline{z}) = \liminf_{r\to 0} \sup_{\overline{z,\underline{z}}\le r} \frac{|f(z) - f(\underline{z})|}{r}\,, \tag{1.14}$$

and put $lip\, f(\underline{z}) = 0$ for $\underline{z}$ isolated.

DEFINITION 1.15. For f Lipschitz, the *pointwise Lipschitz constant*, $\mathrm{Lip}\, f$, is the Borel function,

$$\begin{aligned}\mathrm{Lip}\, f(\underline{z}) &= \limsup_{r\to 0} \sup_{\overline{z,\underline{z}}\le r} \frac{|f(z)-f(\underline{z})|}{r} \\ &= \limsup_{r\to 0} \sup_{\overline{z,\underline{z}}= r} \frac{|f(z)-f(\underline{z})|}{r} \\ &= \limsup_{\overline{z,\underline{z}}\to 0} \frac{|f(z)-f(\underline{z})|}{\overline{z,\underline{z}}}\,, \end{aligned} \tag{1.16}$$

where as above, we put $\mathrm{Lip}\, f(\underline{z}) = 0$ if $\underline{z}$ is isolated.

We have $\mathit{Lip}\, f(z) \le \mathit{lip}\, f(z) \le \mathrm{Lip}\, f(z) \le \mathbf{Lip}\, f$, where in general, the inequalities could be strict.

As in section 0, for $r > 0$, $\underline{z} \in Z$, we put

$$f_{r,\underline{z}} = \frac{f - f(\underline{z})}{r}\,. \tag{1.17}$$

The following proposition is trivial to verify.

PROPOSITION 1.18. *For all Lipschitz functions, f, the following holds:*

i) $\liminf_{r\to 0} \sup_{\overline{z,\underline{z}}=r} |f_{r,\underline{z}}| = 0$, *if and only if $\mathit{Lip}\, f(\underline{z}) = 0$.*
ii) $\liminf_{r\to 0} \sup_{B_r(\underline{z})} |f_{r,\underline{z}}| = 0$, *if and only if $\mathit{lip}\, f(\underline{z}) = 0$.*
iii) $\lim_{r\to 0} \sup_{B_r(\underline{z})} |f_{r,\underline{z}}| = 0$, *if and only if* $\mathrm{Lip}\, f(\underline{z}) = 0$.
iv) *In particular, the set of all functions, f, such that* $\mathrm{Lip}\, f(\underline{z}) = 0$, *forms a subspace of the space of all Lipschitz functions.*

REMARK 1.19. The fibre of the cotangent bundle at a point, $\underline{x} \in \mathbf{R}^n$, can be canonically identified with quotient of the stalk at $\underline{x}$, of the sheaf of differentiable functions, by the subspace consisting of all functions, with a representative, f, such that $\mathrm{Lip}\, f(\underline{x}) = 0$. However, in this characterization, it is not possible to replace differentiable functions by Lipschitz functions, since although Lipschitz functions are differentiable almost everywhere, not every Lipschitz function is differentiable at the given point $\underline{x}$. Thus, we cannot use ii) of Proposition 1.18 to define the fibre at $\underline{z}$ of a generalized cotangent bundle. None-the-less, we will show in Theorems 3.7, 4.38, that given (0.1) and the Poincaré inequality, this difficulty can be circumvented by regarding the cotangent bundle as an L_∞ vector bundle (rather than a topological one). Note that an L_∞ vector bundle does not have canonically defined fibres, but only representatives with fibres defined almost everywhere, such that for any two such representatives, corresponding fibres are canonically identified almost everywhere. Thus, the situation is analogous to that which concerns the pointwise values of an element of an L_p space.

2 Sobolev Spaces and Generalized Upper Gradients

Let Z be a metric space and μ a Borel regular measure on Z. Throughout the paper, we assume that the measure of balls of finite nonzero radius is finite and nonzero.

In this section, for all $1 \le p < \infty$, we define a version the Sobolev space, $H_{1,p}$. Sobolev spaces for general metric spaces were introduced in [H]; compare also [HKo]. The definitions in these references are different from the one employed here, which is more suited to our present purposes.

In the present section, we also define the notion of generalized upper gradient, which plays an important role in the sequel. In particular, we show that at least for the case, $1 < p < \infty$, Sobolev functions can be characterized in terms of this notion.

In [Sh], N. Shanmugalingam defines Sobolev spaces and a corresponding notion of *weak upper gradient*. She has verified that for $p > 1$, her definition, which employs the concept of the p-modulus of a path family, gives rise to the same spaces as ours.

Fix an open set, U and until further notice, write L_p for $L_p(U)$.

For $f \in L_p$, we set

$$|f|_{1,p} = |f|_{L_p} + \inf_{\{g_i\}} \liminf_{i\to\infty} |g_i|_{L_p} \,, \tag{2.1}$$

where the inf is taken over all sequences, $\{g_i\}$, for which there exists a sequence, $f_i \xrightarrow{L_p} f$, such that g_i is an upper gradient for f_i, for all i. Note that by Proposition 2.20 below, we could as well require that each of the functions, f_i, be bounded. It is trivial to check that set of elements, $f \in L_p$, on which $|f|_{1,p} < \infty$ is a subspace, for which this expression defines a norm. We call $(|f|_{1,p} - |f|_{L_p})^p$ the *upper gradient p-energy* of f.

DEFINITION 2.2. For $p \ge 1$, the Sobolev space, $H_{1,p} = H_{1,p}(U)$, is the subspace of L_p consisting of functions, f, for which $|f|_{1,p} < \infty$, equipped with the norm $|\cdot|_{1,p}$.

REMARK 2.3. If Z is rectifiably connected, then the Sobolev space, $H_{1,p}$, remains unchanged as a normed linear space, if the given metric on Z is replaced by the canonically associated length space metric; see Remark 1.4.

Let $0 \to H_{1,p} \xrightarrow{i} L_p$ denote the natural map.

REMARK 2.4. It is clear that the map, i, is a norm nonincreasing injection such that $i(\overline{B_r(0)})$ is closed in L_p. Here, $\overline{B_r(0)} = \{v \in H_{1,p} \mid |v|_{1,p} \le r\}$.

As follows immediately from the definition, the upper gradient p-energy is lower semicontinuous with respect to convergence in L_p, i.e. Rellich's theorem holds for the upper gradient p-energy.

Theorem 2.5. *Let $\{f_i\}$ be a bounded sequence in $H_{1,p}$ such that $i(f_i) \xrightarrow{L_p} f$. Then $f = i(f_\infty)$, where f_∞ satisfies*

$$\liminf_i |f_i|_{1,p} \ge |f_\infty|_{1,p} \,. \tag{2.6}$$

Theorem 2.5 allows us to deduce:

Theorem 2.7. *The space, $H_{1,p}$, is complete.*

Proof. Let $\overline{H}_{1,p}$ denote the completion of $H_{1,p}$. The map, i, extends uniquely to a norm nonincreasing map, $\overline{i} : \overline{H}_{1,p} \to L_p$.

We claim that $\overline{i}$ is an injection. To see this, let $\{u_j\}$ be a Cauchy sequence in $H_{1,p}$ such that $i(u_j) \xrightarrow{L_p} 0$. It suffices to show that $u_j \xrightarrow{H_{1,p}} 0$.

Assume the contrary. It follows that $\lim_{j\to\infty} |u_j|_{1,p} = c > 0$. Fix j so large that we have $|u_j - u_k|_{1,p} < c/2$, for all $k \ge j$. Thus, $\limsup_k |u_j - u_k|_{1,p} \le c/2$. Letting $k \to \infty$, we get $i(u_j - u_k) \xrightarrow{L_p} i(u_j)$. Since $\lim_{j\to\infty} |u_j|_{1,p} = c > 0$, $\limsup_k |u_j - u_k|_{1,p} \le c/2$, we contradict Theorem 2.5.

Now let $\{f_\ell\}$ be a sequence in $H_{1,p}$, such that $f_\ell \xrightarrow{\overline{H}_{1,p}} \overline{f}_\infty$. Since in particular, $\{f_\ell\}$ is bounded in $H_{1,p}$, it follows from Remark 2.4 that $\overline{i}(\overline{f}_\infty) = i(f_\infty)$, for some $f_\infty \in H_{1,p}$. Since $\overline{i}$ extends i, we also have $\overline{i}(\overline{f}_\infty) = \overline{i}(f_\infty)$. But $\overline{i}$ is an injection, so $f_\infty = \overline{f}_\infty$. This suffices to complete the proof. □

DEFINITION 2.8. The function, $g \in L_p$ is a *generalized upper gradient* for $f \in L_p$, if there exist sequences, $f_i \xrightarrow{L_p} f$, $g_i \xrightarrow{L_p} g$, such that g_i is an upper gradient for f_i, for all i.

Clearly, for any g as above, we have $|f|_{1,p} \le |f|_{L_p} + |g|_{L_p}$. Moreover, the set of generalized upper gradients is a closed convex subset of L_p. A priori, for a given $f \in H_{1,p}$, this subset could be empty; see however Theorem 2.10 below, for the case $1 < p < \infty$.

DEFINITION 2.9. For fixed p, a *minimal generalized upper gradient* for f is a generalized upper gradient g_f, such that $|f|_{1,p} = |f|_{L_p} + |g_f|_{L_p}$.

Theorem 2.10. *For all $1 < p < \infty$ and $f \in H_{1,p}$ there exists a minimal generalized upper gradient, g_f, which is unique up to modification on subsets of measure zero.*

Proof. We can choose a suitable diagonal sequence, $f_i \xrightarrow{L_p} f$, and for all i, an upper gradient, g_i for f_i, such that $|f|_{1,p} = |f|_{L_p} + \lim_{i\to\infty} |g_i|_{L_p}$. From the uniform convexity of L_p, for $1 < p < \infty$, it follows that there exists g such that $g_i \xrightarrow{L_p} g$. Clearly, $g = g_f$ is an upper gradient for f. Again by uniform convexity, it follows that g is independent of the particular choice of sequences, $\{f_i\}, \{g_i\}$. □

DEFINITION 2.11. Let $1 \le p \le \infty$. A sequence, $f_i \xrightarrow{L_P} f$, converges to $f \in H_{1,p}$ in the *relaxed topology* of $H_{1,p}$, if

$$\lim_{i\to\infty} |f_i|_{1,p} = |f|_{1,p}\,, \tag{2.12}$$

By using Theorem 2.5 and the uniform convexity of L_p, for $1 < p < \infty$, we get:

PROPOSITION 2.13. *Let $1 < p < \infty$ and let $f_i \xrightarrow{L_p} f$. Then the following conditions are equivalent:*

i) *f_i converges to f in the relaxed topology of $H_{1,p}$.*
ii) $g_{f_i} \xrightarrow{L_p} g_f$.
iii) *There exist generalized upper gradients, g_i for f_i, such that $|f_i|_{L_p} + |g_i|_{L_p} \to |f|_{1,p}$.*
iv) *There exist generalized upper gradients, g_i for f_i, such that $g_i \xrightarrow{L_p} g_f$.*

REMARK 2.14. Let ρ_1, ρ_2 be metrics on Z with $\rho_1 \le \rho_2$. Then it is easy to see that $g_f^{\rho_2} \le g_f^{\rho_1}$, where $g_f^{\rho_1}, g_f^{\rho_2}$, denote the minimal upper gradients with respect to ρ_1, ρ_2, respectively.

REMARK 2.15. If for example $Z = \mathbf{R}^n$, then the (mutually equivalent) conditions in Proposition 2.13 imply $f_i \xrightarrow{H_{1,p}} f$. More generally, this follows in those cases in which the norm on $H_{1,p}$ is uniformly convex; compare Theorem 4.48 and the discussion which follows.

REMARK 2.16. Let $p = 1$. As in the case, $1 < p < \infty$, we can choose a sequence, $f_i \xrightarrow{L_p} f$, and for all i, an upper gradient, g_i for f_i, such that $|f|_{1,p} = |f|_{L_p} + \lim_{i\to\infty} |g_i|_{L_p}$. From the compactness of the space of Radon measures, it follows that there exists a Radon measure, g_f, such that $\{g_i\}$ as in (2.12) converges to g_f in the sense of Radon measures. However, such a g_f might not be unique and, as a consequence of oscillations, the mass of g_f could be smaller than $|f|_{1,1} - |f|_{L_1}$. The proper context for discussing these issues is that of *functions of bounded variation*, but we will not pursue that discussion here.

Note that if $W \subset U$ is open, then $g \,|\, W$ is a generalized upper gradient for $f \,|\, W$.

Proposition 2.17. *Let $f : U \to \mathbf{R}$ and let g_U be a generalized upper gradient for f on U. Let $W \subset U$ be open and let g_W be a generalized upper gradient for $f \,|\, W$. Then there exists a generalized upper gradient, $\underline{g}_U$, for f on U, such that $\underline{g}_U(z) = g_W(z)$, for μ-a.e. $z \in W$, and $\underline{g}_U(z) = g_U(z)$, for μ-a.e. $z \in (U \setminus W)$.*

Proof. Since W is a countable union of bounded open sets, it clearly suffices to assume that W itself is bounded. Let $f_{U,i} \xrightarrow{L_p} f$, $g_{U,i} \xrightarrow{L_p} g_U$, be sequences as in Definition 2.8. Let $f_{W,i} \xrightarrow{L_p} f$, $g_{W,i} \xrightarrow{L_p} g_W$, be the corresponding sequences for $f \,|\, W$. Fix $\eta > 0$ and let $W_\eta \subset W$ denote the set of points at distance $\geq \eta$ from ∂W. Let $\phi : U \to [0,1]$ be a Lipschitz function with $\operatorname{supp} \phi \subset W$, such that $\phi \,|\, W_\eta \equiv 1$.

Define $f_i : U \to \mathbf{R}$ by $f_i = \phi f_{W,i} + (1-\phi) f_{U,i}$. Then $f_i \xrightarrow{L_p} f$. Since on W, we have $f_i = (1-\phi)(f_{U,i} - f_{W,i}) + f_{W,i}$, it follows from Lemma 1.7 that for all $\epsilon > 0$, an upper gradient for $f_i \,|\, W$ is provided by the function, $\operatorname{Lip}(1-\phi)\,(|f_{U,i} - f_{W,i}| + \epsilon) + ((1-\phi)+\epsilon)(g_{U,i} + g_{W,i}) + g_{W,i}$. On $U \setminus \operatorname{supp} \phi$, the function, g_{U_i}, is an upper gradient for $f_i \,|\, U \setminus \operatorname{supp} \phi$. Extend each of these functions to all of U by setting them equal to 0 on the $U \setminus W$, respectively, $\operatorname{supp} \phi$. By Proposition 1.6 the maximum of the extended functions is an upper gradient for f_i on U. Let $i \to \infty$, then $\epsilon \to 0$, and finally, $\eta \to 0$. Since, W is bounded and μ is a Borel regular measure for which the measure of bounded sets is finite, our claim easily follows. □

Theorem 2.18. *If g is any generalized upper gradient for f, then*

$$g_f \leq g \qquad (\mu\text{-a.e.})\,. \tag{2.19}$$

Proof. Assume that there exists a bounded set, A, with $\mu(A) > 0$, such that $g(\underline{z}) < g_f(\underline{z}) - \epsilon$, for all $\underline{z} \in A$. Choose a sequence of open sets, $W_i \supset A$, such that $\mu(W_i \setminus A) \to 0$. By applying Proposition 2.17 on each set, W_i and letting $i \to \infty$, we contradict the assumption that g_f is a minimal generalized upper gradient. □

Proposition 2.20 (Truncation). *If U is bounded then for all $c \leq \infty$, we have*

$$\big|\min(f,c)\big|_{1,p} - \big|\min(f,c)\big|_{L_p} \leq |f|_{1,p} - |f|_{L_p}\,. \tag{2.21}$$

Proof. This is a direct consequence of the corresponding fact for upper gradients which is trivially verified. □

Proposition 2.22. *If g is a generalized upper gradient for f and $A = f^{-1}(c)$, for some constant c, then there exists a generalized upper gradient, $\underline{g}$, for f such that $\underline{g} = g$, μ-a.e. on $U \setminus A$ and $\underline{g} = 0$, μ-a.e. on A.*

Proof. We can write U as a union of bounded open sets. Since the set of generalized upper gradients of f is a closed subset of L_p, it suffices to assume that U is bounded. Let $f_i \xrightarrow{L_p} f$, $g_i \xrightarrow{L_p} g$, be sequences as in Definition 2.8. For all $\epsilon > 0$, define

$$f_{i,\epsilon}(z) = \begin{cases} f_i(z) + \epsilon & \text{if } f_i(z) \le c - \epsilon\,, \\ c & \text{if } c - \epsilon \le f_i(z) \le c + \epsilon\,, \\ f_i(z) - \epsilon & \text{if } c + \epsilon \le f_i(z)\,. \end{cases} \tag{2.23}$$

Then $\lim_{\epsilon \to 0} \lim_{i \to \infty} f_{i,\epsilon} = f$, in L_p.

Since, μ is Borel regular, there exists a sequence, $\{C_{i,j}\}$, closed subsets of $f_{i,\epsilon}^{-1}(c)$, such that $\lim_{j \to \infty} \mu(C_{i,j}) = \mu(f_{i,\epsilon}^{-1}(c))$. Let $g_{i,\epsilon,C_{i,j}}$ denote the function such that $g_{i,\epsilon,C_{i,j}} = g_i$ on $U \setminus C_{i,j}$ and $g_{i,\epsilon,C_{i,j}} = 0$ on $C_{i,j}$. It is straightforward to check that $g_{i,\epsilon,C_{i,j}}$ is an upper gradient for $f_{i,\epsilon}$.

For fixed $\epsilon > 0$, we have $\lim_{i \to \infty} \mu(A \setminus f_{i,\epsilon}^{-1}(c)) = 0$. Let $j \to \infty$, then $i \to \infty$ and finally, $\epsilon \to 0$. By taking a suitable diagonal sequence, the proof is easily completed. □

Corollary 2.24. *For $1 < p < \infty$, the subspace of $H_{1,p}$ consisting of those functions which are essentially bounded is dense in $H_{1,p}$.*

Corollary 2.25. *Let $1 < p < \infty$. If $f_1, f_2 \in H_{1,p}$ and $f_1 \,|\, A = f_2 \,|\, A$, for some Borel set, $A \subset U$, then $g_{f_1}(z) = g_{f_2}(z)$, for μ-a.e. $z \in A$.*

Proof. Since, $f_1 = (f_1 - f_2) + f_2$, we get $g_{f_1} \le g_{(f_1 - f_2)} + g_{f_2}$, which, by Proposition 2.22, gives $g_{f_1}(z) \le g_{f_2}(z)$, for μ-a.e. $z \in A$. The reverse inequality follows similarly. □

Corollary 2.26. *If $f_1, f_2 \in H_{1,p}$ then either $g_{\min(f_1,f_2)}(z) = g_{f_1}(z)$, or $g_{\min(f_1,f_2)}(z) = g_{f_2}(z)$, for μ-a.e. z.*

Proof. By Proposition 1.5, we have $\min(f_1, f_2) \in H_{1,p}$. Thus, our assertion follows from Corollary 2.25. □

Remark 2.27. Proposition 2.17 enables us to extend the definition of minimal generalized upper gradients to functions which are only locally in $H_{1,p}$. Note also that when the definition is so extended, the *pointwise* μ-a.e. minimizing property of minimal generalized upper gradients implies that g_f remains unchanged if the measure, μ is replaced by a measure μ', provided μ, μ' are mutually absolutely continuous and $L_p(Z, \mu) = L_p(Z, \mu')$.

REMARK 2.28. If the set of generalized upper gradients for $f \in H_{1,1}$ is nonempty, then by a replacement argument based on Proposition 2.17 (similar to the one given in the proof of Theorem 2.18) it follows that there exists a unique smallest generalized upper gradient, $\underline{g}_f$, such that for all generalized upper gradients, g, we have $\underline{g}_f(z) \leq g(z)$, for μ-a.e. $z \in Z$. However, even in this case, this argument does not show that $|f|_{1,1} = |f|_{L_1} + |\underline{g}_f|_{L_1}$, i.e. that $\underline{g}_f$ is a minimal generalized upper gradient.

REMARK 2.29. So far we have not addressed the issue of the possible dependence of g_f on the choice of p. For example, U bounded and $f \in H_{1,p}$, implies $f \in H_{1,p'}$, for $1 < p' < p$. If we write $g_{f,p}$ to indicate the dependence on p, then clearly, $g_{f,p'} \leq g_{f,p}$. Note that although the generalized upper gradient, $g_{f,p'}$, is in L_p, it is not clear that there exists, $f_i \xrightarrow{L'_p} f$, $g_i \xrightarrow{L_p} g_{f,p'}$. If this holds, then a truncation argument implies $g_{f,p} = g_{f,p'}$. At least in the case in which Z the measure is doubling and a Poincaré inequality of type $(1,p')$ holds, we do have $g_{f,p} = g_{f,p'}$; see Corollary 6.38.

Let U be open and let U_η be as in Proposition 2.17. Let $\mathcal{K}(U)$ denote the subset of $H_{1,p}(U)$ consisting of those functions, k, for which there exists $\eta > 0$, such that $i(k)$, the image of k, in L_p, has (a representative) with support in U_η.

DEFINITION 2.30. The Sobolev space, $\overset{\circ}{H}_{1,p}(U) \subset H_{1,p}(U)$, is the closure in $H_{1,p}(U)$ of the space $\mathcal{K}(U)$.

REMARK 2.31. Note that if $k \in \mathcal{K}(W)$ and $W \subset U$, then by putting $k \,|\, U \backslash W \equiv 0$, we can regard $k \in \mathcal{K}(U)$. We denote the extended function by k as well. By an argument similar to that given in the proof of Lemma 2.17, we can also regard $\overline{k} \in \overset{\circ}{H}_{1,p}(W)$ as an element of $\overset{\circ}{H}_{1,p}(U)$. We also denote the extended function by $\overline{k}$.

3 The Vitali Covering Theorem and Asymptotic Generalized Linearity

In this section we give a partial generalization of Rademacher's theorem on the almost everywhere differentiability of Lipschitz functions on Euclidean spaces, under the assumptions that the pair (Z,μ) satisfies the Vitali covering theorem. We recall that this is implied by (0.1); see Chapter 2 of [Ma] and the references therein for additional examples. We also observe that absent any assumptions on (Z,μ), functions which almost satisfy the conditions of approximate generalized linearity, satisfy a reverse Poincaré inequality as well. However, for the application to asymptotically generalized linear functions, it is neccessary to assume (0.1).

Throughout this section, we fix some p with $1 < p < \infty$. However, since our assertions are to the effect that something holds μ-a.e., it follows immediately that these assertions actually hold μ-a.e., for all values of p lying in a prescribed countable subset of $(1,\infty)$. It will turn out that under the assumption that a Poincaré inequality of type $(1,p)$ holds, our assertions will hold μ-a.e., for all values $p' \in [p,\infty)$; see Remark 2.29 and Corollary 6.38.

Fix $\underline{z} \in Z$, $r > 0$, and let $\overline{B}_r(\underline{z})$ denote the closed ball of radius r and center $\underline{z}$. Let $\overline{k}_r$ denote a function in $\overset{\circ}{H}_{1,p}\,(B_r(\underline{z}))$.

Definition 3.1. Let $1 < p < \infty$. The function, f, is *asymptotically p-harmonic* at $\underline{z} \in Z$, if

$$\lim_{r\to 0}\left(\fint_{\overline{B}_r(\underline{z})} (g_f)^p\,d\mu - \inf_{\overline{k}_r} \fint_{\overline{B}_r(\underline{z})} (g_{f+\overline{k}_r})^p\,d\mu \right) = 0\,. \tag{3.2}$$

Note that in Definition 3.1, we could as well have used functions, $k_r \in \mathcal{K}(B_r(\underline{z}))$.

It is clear that the property of being asymptotically harmonic at $\underline{z}$ is independent of the particular choice of minimal generalized upper gradient (and could actually be reformulated to include the case $p = 1$, where we can not be certain that such exist). In order to avoid such dependence in our next definition, it is convenient to make the following convention whenever the Vitali covering theorem holds.

Convention. If the Vitali covering theorem holds, then $(g_f)^p$ will always denote the *precise representative* of $(\hat{g}_f)^p$, where $\hat{g}_f$ is any minimal generalized upper gradient; see p. 46 of [EG]. In particular, the function, g_f, is independent of the choice of $\hat{g}_f$ and vanishes at all points which are not Lebesgue points of $(g_f)^p$.

With the above convention, we have $g_{af} = |a| g_f$, for all $a \in \mathbf{R}$. Moreover, if $\underline{z}$ is a Lebesgue point of $(g^p_{f_1})$ and $(g_{f_2})^p$, then

$$g_{f_1+f_2}(\underline{z}) \le g_{f_1}(\underline{z}) + g_{f_2}(\underline{z})\,. \tag{3.3}$$

Finally, if g is a generalized upper gradient of f and $\underline{z}$ is a Lebesgue point of of $(g)^p$ and of $(g_f)^p$, then

$$g_f(\underline{z}) \le g(\underline{z})\,. \tag{3.4}$$

Definition 3.5. The function, f, is *asymptotically generalized linear with respect to* g_f at $\underline{z} \in Z$ if f is asymptotically p-harmonic at $\underline{z}$, and the point,

$\underline{z}$, is a Lebesgue point of $(g_f)^p$.

Remark 3.6. Note that g_f is also the minimal generalized upper gradient for the function, $f_{r,\underline{z}}$, with respect to the rescaled metric, $r^{-1}\rho$, where ρ is the metric (i.e. distance) on Z. If we consider the family of functions obtained by restricting each function, $f_{r,\underline{z}}$, to the ball, $B_r(\underline{z})$, equipped with the rescaled metric $r^{-1}\rho$, then the condition of asymptotic generalized linearity has an obvious reformulation in terms of the behavior as $r \to 0$, of this family of functions.

Theorem 3.7 (Asymptotic generalized linearity; Rademacher 1). *If* (Z,μ) *satisfies, the Vitali covering theorem* (*in particular, if* μ *satisfies the doubling condition,* (0.1)) *and* $f : Z \to \mathbf{R}$ *is Lipschitz, then* f *is asymptotically generalized linear, for* μ-a.e. $z \in Z$.

Proof. Assume that there exists $A \subset Z$, of positive measure, such that for some $\epsilon > 0$ and all $\underline{z} \in A$, there is a positive sequence, $r_j(\underline{z}) \to 0$, such that $\underline{z}$ is a Lebesgue point of $(g_f)^p$ and for some function, $k_{r_j(\underline{z})} \in \mathcal{K}(B_{r_j}(\underline{z}))$,

$$⨍_{\overline{B}_{r_j(\underline{z})}(\underline{z})} (g_f)^p \, d\mu \geq \epsilon + ⨍_{\overline{B}_{r_j(\underline{z})}(\underline{z})} (g_{f+k_{r_j}})^p \, d\mu \,. \tag{3.8}$$

We can assume without loss of generality that the function, $f + k_{r_j(\underline{z})}$, satisfies, $\max f + k_{r_j(\underline{z})} \leq \max_{B_{r_j(\underline{z})}(\underline{z})} f$, $\min f + k_{r_j(\underline{z})} \geq \min_{\overline{B}_{r_j(\underline{z})}(\underline{z})} f$; see Proposition 2.20. Thus, on $\overline{B}_{r_j(\underline{z})}(\underline{z})$, we have $|f-(f+k_{r_j(\underline{z})})|_{L_\infty} \leq 2Lr_j(\underline{z})$, where L is the Lipschitz constant of f.

Since μ satisfies the Vitali covering theorem, for every positive integer, i, we can find a covering of almost all of A, by a collection, $\{\overline{B}_{r_{i,k}}(z_k)\}$, of mutually disjoint closed balls, such that for all k, we have $z_k \in A$, $r_{i,k} < i^{-1}$, and $r_{i,k} = r_j(z_k)$, for some j. Let $f_i \in H_{1,p}(U)$ be the function whose restriction to each $B_{r_{i,k}}(z_k)$ coincides with $f + k_{r_{i,k}}$ and which is equal to f elsewhere. Clearly, $f_i \xrightarrow{L_\infty} f$ and by construction, together with Corollary 2.25, $\liminf_i |f_i|^p_{1,p} \leq |f|^p_{1,p} - \epsilon\mu(A)$. This contradicts Theorem 2.5 (Rellich's Theorem.) Hence, $\mu(A) = 0$, which completes the proof. □

Remark 3.9. Note that for the proof of Theorem 3.7 to go through, it would suffice to assume that $f \in H_{1,p}$ and in addition, that f is continuous.

We now show that if f is asymptotically generalized linear at $\underline{z}$, then f, satisfies a reverse Poincaré inequality on sufficiently small balls $B_r(\underline{z})$.

For U open, let U_η be as in Lemma 1.7.

LEMMA 3.10 (Reverse Poincaré inequality). *Fix $0 \le \delta < \psi \le 1$. Let $f \in H_{1,p}(U)$ satisfy*

$$\left(\int_{U\setminus U_{2\eta}} (g_f)^p \, d\mu\right)^{1/p} \le \delta\left(\int_U (g_f)^p \, d\mu\right)^{1/p}, \tag{3.11}$$

and assume that for all $k \in \mathcal{K}(U)$,

$$\psi\left(\int_U (g_f)^p \, d\mu\right)^{1/p} \le \left(\int_U (g_{f+k})^p \, d\mu\right)^{1/p}. \tag{3.12}$$

Then

$$\left(\int_U (g_f)^p \, d\mu\right)^{1/p} \le \frac{1}{\eta(\psi-\delta)}\left(\int_{U_\eta\setminus U_{2\eta}} |f|^p \, d\mu\right)^{1/p}. \tag{3.13}$$

Proof. For all $\eta > 0$, there exists a Lipschitz function, $\phi : U \to [0,1]$, such that $\phi \,|\, U_{2\eta} \equiv 1$, $\operatorname{supp}\phi \subset U_\eta$, $|\mathrm{Lip}\,(1-\phi)|_{L_\infty} \le \eta^{-1}$. By Lemma 1.7, for all $\epsilon > 0$, the function, $\mathrm{Lip}\,(1-\phi)\,(|f|+\epsilon) + (1-\phi+\epsilon)g_f$, is a generalized upper gradient for the function $(1-\phi)f$. By letting $\epsilon \to 0$ and using (3.11), (3.12), the proof is easily completed. □

Although (0.1) was not required in Lemma 3.10, we do make this assumption in Theorem 3.14 below.

Theorem 3.14. *Let (Z,μ) satisfy* (0.1). *Let $f \in H_{1,p}$ be asymptotically generalized linear at $\underline{z}$. If $g_f(\underline{z}) > 0$, then for all $r > 0$ sufficiently small,*

$$r\left(\int_{B_r(\underline{z})} (g_f)^p \, d\mu\right)^{1/p} \le 2^{\frac{2\kappa+4}{p}}\left(\int_{B_{\frac{2}{3}r}(\underline{z})\setminus B_{\frac{1}{3}r}(\underline{z})} |f|^p \, d\mu\right)^{1/p}. \tag{3.15}$$

In particular, $\mathrm{lip}\, f(\underline{z}) > 0$.

Proof. This follows from Lemma 3.10 by taking, $U = B_r(\underline{z})$, $\eta = \frac{1}{3}r$, $\delta = 1 - 2^{-2\kappa/p}$, $\psi = 1 - 2^{-(2\kappa+2p)/p}$ and using (0.1). □

In concert with with Theorem 4.8, Theorem 3.14 plays an important role in the proof of the uniqueness statement for our generalization of Rademacher's theorem, and in our discussion of the generalized cotangent bundle, T^*Z, as well; see Theorem 4.38.

4 The Poincaré Inequality and Differentials of Lipschitz Functions

The classical Rademacher theorem for Lipschitz functions on Euclidean space asserts that at almost all points, not only are such functions asymptotically (in fact, infinitesimally) linear, but in addition, the asymptotic linear approximation is the same on all scales. In this section, given the doubling condition and Poincaré inequality, we prove a statement which when specialized to the Euclidean case, implies this uniqueness; see Theorem 4.38. As a consequence, we show that such a space has a finite dimensional L_∞ vector bundle, the generalized cotangent bundle, of which the differentials of Lipschitz functions are sections. We also consider the induced maps on cotangent bundles for (suitable) Lipschitz maps between spaces satisfying our assumptions and, under suitable additional assumptions, for quasiconformal homeomorphisms.

The existence of the finite dimensional generalized cotangent bundle has strong implications. For example, the arguments and conclusions of [HeKM], developed there in the context of p-admissible weights, apply in our situation as well.

In the present section (apart from Conjecture 4.65) we consider neither measured Gromov-Hausdorff convergence, nor tangent cones, nor in particular, limit functions, $f_{0,\underline{z}}$, on tangent cones. These are discussed in sections 9–14.

Throughout this section, we assume (0.1).

For $f \in L_1$, set

$$\frac{1}{\mu(W)}\int_W f\,d\mu = \fint_W f\,d\mu \tag{4.1}$$

and

$$f_{\underline{z},r} = \fint_{B_r(\underline{z})} f\,d\mu\,. \tag{4.2}$$

The quantity, $f_{\underline{z},r}$, should not be confused with the functions, $f_{r,\underline{z}}$, which appear in section 0; see also (1.17).

We say that Z satisfies a *weak Poincaré inequality of type* $(1,p)$, if for all $r' > 0$, there exists $1 \le \Lambda < \infty$ and $C = C(p, r')$, such that for all $r \le r'$, and all upper gradients, g, for f,

$$\big(|f - f_{\underline{z},r}|\big)_{\underline{z},r} \le Cr\big((g^p)_{\underline{z},\Lambda r}\big)^{1/p}\,. \tag{4.3}$$

By Hölder's inequality, if (4.3) holds for some p, it holds for all p', with $p \le p' < \infty$. Also, according to [HeKo3], if Z is *proper* (i.e. closed balls of finite radius are compact) and quasi-convex, then (4.3) holds (for all measurable functions f) if and only if it holds for all Lipschitz functions f.

If (0.1) holds and in addition, balls in Z satisfy a chain condition with constant $0 < M < \infty$, then by [J], [HKo], (4.3) implies the Poincaré-Sobolev inequality,

$$\big((|f - f_{\underline{z},r}|^{\chi p})_{\underline{z},r}\big)^{1/\chi p} \le \tau r\big((g^p)_{\underline{z},r}\big)^{1/p}\,, \tag{4.4}$$

where $\chi = \chi(\kappa, C) > 1$, $\tau = \tau(\kappa, C, M, \Lambda)$. In particular, there exists M such that if Z is a length space, then Z satisfies a chain condition with constant M. According to [DSe2], for Z complete, (0.1), (4.3) imply that Z is $\lambda(\kappa, C)$-quasi-convex, and hence, $\lambda(\kappa, C)$-quasi-isometric to a length space metric; see also section 17 and [Se5]. Hence, it follows that (0.1), (4.3) imply (4.4), with constant $\tau = \tau(\kappa, C, \Lambda)$.

It is also observed in [HKo], that if for some $\psi > 0$, there exists z', such that $\overline{z', \underline{z}} \geq (1+\psi)r$, then (0.1), (4.3) imply the following Sobolev inequality, for all $f \in \overset{\circ}{H}_{1,p}\,(B_r(\underline{z}))$.

$$\left(\fint_{B_r(\underline{z})} |f(z)|^{\chi p}\, d\mu \right)^{1/\chi p} \leq c(\kappa, p, \psi) \tau^{1/p} r \left(\fint_{B_r(\underline{z})} (g)^p\, d\mu \right)^{1/p}; \quad (4.5)$$

compare [LiS]. Since in the proof, one applies (4.22) (with $\underline{z}$ of (4.22) replaced by a suitable point, $z'' \in (B_{(1+\psi)r}(\underline{z}) \setminus B_r(\underline{z})))$ the chain condition does not enter this discussion.

Remark 4.6. It is an immediate consequence of the definitions that the validity of any of (4.3)–(4.5) for all f and all upper gradients, g, implies the validity of the corresponding inequalities for all f and all *generalized* upper gradients g. This remark will be used from now on without further mention.

Let $U_\eta \subset U$ be as in Lemma 3.10. Let i^* denote the restriction map from functions on U to functions on U_η. If $U \subset B_d(\underline{z})$, it follows from (0.1) that for all $0 < s \leq \eta$, there exists $N_1 = N_1(\kappa, d, s)$, $N_2 = N_2(\kappa, \Lambda)$ and coverings, $U_\eta \subset \cup_{i=1}^{N_1} B_s(z_i) \subset \cup_{i=1}^{N_1} B_{\Lambda s}(z_i) \subset U$, such that the covering, $\{B_{\Lambda s}(z_i)\}$, has multiplicity $\leq N_2$. Given such coverings and $f \in L_p(U)$, we put

$$\phi(f) = \big((\mu(B_s(z_1)))^{1/p} f_{z_1,s}, \cdots, (\mu(B_s(z_{N_1})))^{1/p} f_{z_{N_1},s} \big)\,, \quad (4.7)$$

and equip $\mathbf{R}^n$ with the norm, $|(a_1, \ldots, a_n)|_p^p = |a_1|^p + \cdots + |a_n|^p$.

Theorem 4.8. *Let (Z, μ) satisfy (0.1) and (4.3). If*

$$\eta^p \int_U g_f^p\, d\mu \leq K^p \int_{U_\eta} |f|^p\, d\mu \quad (4.9)$$

and

$$0 < s \leq \lambda \eta\,, \quad (4.10)$$

where $\lambda = \min(1, (4\tau K N_2^{1/p})^{-1})$, then

$$\begin{aligned} \int_{U_\eta} |f|^p\, d\mu &\leq 2^{p+1} \sum_{i=1}^{N_1} |f_{z_i,s}|^p \mu(B_s(z_i)) \\ &= 2^{p+1} |\phi(f)|_p^p\,. \end{aligned} \quad (4.11)$$

In particular, if $\mathcal{V}$ is a space of continuous functions such that (4.9) *holds for all f in $\mathcal{V}$, then* $\dim(\mathcal{V}) \leq N_1$. *Moreover, if $g_f \equiv 0$ implies $f \equiv c$, for some constant, c, then* $\dim(\mathcal{V}) \leq N_1$.

Proof. We have

$$\int_{U_\eta} |f|^p d\mu \leq \sum_i \int_{B_s(z_i)} |f|^p d\mu$$
$$\leq 2^p \sum_i \int_{B_s(z_i)} |f - f_{z_i,s}|^p d\mu + 2^p \sum_i \int_{B_s(z_i)} |f_{z_i,s}|^p d\mu\,, \quad (4.12)$$

and by (4.4), (4.9),

$$\sum_i \int_{B_s(z_i)} |f - f_{z_i,s}|^p d\mu \leq N_2 \tau^p s^p \int_U g_f^p d\mu$$
$$\leq \tau^p \lambda^p K^p N_2 \int_{U_\eta} |f|^p d\mu\,. \quad (4.13)$$

For $0 < \lambda \leq \min(1, (4\tau K N_2^{1/p})^{-1})$, from (4.12), (4.13) and (0.1), we obtain (4.11). □

In addition to the above result (compare [CoMi1]) we will need a result of standard type concerning the pointwise behavior of Sobolev functions off subsets of small measure.

Theorem 4.14. *Let (Z, μ), satisfy* (0.1), (4.3). *Let $f : B_d(\overline{z}) \to \mathbf{R}$, and let g be a generalized upper gradient for f. Then for all $K > 0$, there exists a collection of balls, $\{B_{6r_i}(z_i)\}$, satisfying, $B_{r_i}(z_i) \subset B_d(\overline{z})$,*

$$\sum_i \left(\tfrac{r_i}{4d}\right)^\kappa \leq \tfrac{1}{K}\,, \quad (4.15)$$

and

$$\sum_i \frac{\mu(B_{6r_i}(z_i))}{\mu(B_d(\overline{z}))} \leq \frac{8}{K}\,, \quad (4.16)$$

such that if $\underline{z} \in B_d(\overline{z}) \setminus \cup_i B_{6r_i}(z_i)$, and $r \leq \overline{\underline{z}, \partial B_d(\overline{z})}$, then

$$((g^p)_{\underline{z},r})^{1/p} < K^{1/p} \left((g^p)_{\overline{z},d}\right)^{1/p}\,, \quad (4.17)$$

and for all $j > 1$,

$$|f_{\underline{z},2^{-j}r}| \leq \left(\tfrac{4d}{r}\right)^\kappa (|f|)_{\overline{z},d} + rK^{1/p} 2^{\kappa+1} \tau((g^p)_{\overline{z},d})^{1/p}\,. \quad (4.18)$$

Moreover, if $\underline{z}_1, \underline{z}_2 \in B_d(\overline{z}) \setminus \cup_i B_{6r_i}(z_i)$, and $\overline{\underline{z}_1, \partial B_d(\overline{z})} \geq 4\Lambda \overline{\underline{z}_1, \underline{z}_2}$, then

$$\left|f(\underline{z}_1) - f(\underline{z}_2)\right| \leq K^{1/p}\, 2^{2\kappa+2} \tau((g^p)_{\overline{z},d})^{1/p} \overline{\underline{z}_1, \underline{z}_2}\,. \quad (4.19)$$

Remark 4.20. The crucial role in the proof of Theorem 4.14 is played by the following estimate, which is an easy consequence of the weak Poincaré inequality (4.3).

$$\begin{aligned}|f_{\underline{z},\frac{1}{2}r} - f_{\underline{z},r}| &\leq \fint_{B_{\frac{1}{2}r}(\underline{z})} |f - f_{\underline{z},r}|\, d\mu \\ &\leq 2^{\kappa}\big(|f - f_{\underline{z},r}|\big)_{\underline{z},r} \\ &\leq 2^{\kappa}\tau r\big((g^p)_{\underline{z},\Lambda r}\big)^{1/p} . \end{aligned} \tag{4.21}$$

In the proof of Theorem 4.14, the inequality, (4.21), is applied to the balls, $B_{2^{-j}t}(\underline{z})$, where $1 \leq j < \infty$, and $\underline{z} = \underline{z}_1$ or $\underline{z} = \underline{z}_2$; see [EG] or [KiM] for details.

If $\underline{z}$ is a Lebesgue point of f, we get the "telescope estimate" for $|f_{\underline{z},r}|$,

$$|f_{\underline{z},r}| \leq |f(\underline{z})| + 2^{\kappa+1}\tau r\, M_p g\, , \tag{4.22}$$

where $M_p h$ denotes the *maximal function*,

$$M_p h(\underline{z}) = \sup_r ((|h|^p)_{\underline{z},r})^{1/p} . \tag{4.23}$$

This estimate plays a key role in section 5; see Lemma 5.27. In the context of Theorem 4.14, it seems more suggestive to speak of "microscope" estimates; see, e.g. (4.19).

From Theorem 4.14 and a standard argument, we get the following result; for a significantly stronger result, see Theorem 5.3.

Theorem 4.24. *If* (Z,μ) *satisfy* (0.1), *and* (4.3) *for some for* $1 \leq p < \infty$, *then the subspace of locally Lipschitz functions is dense in* $H_{1,p}$. *If* $p = 1$, *such functions are dense in* $H_{1,1}$, *with respect to the relaxed topology of* $H_{1,1}$. *Moreover, the subspace of* $\mathcal{K}$ *consisting of those functions which are of locally Lipschitz is dense in* $\overset{\circ}{H}_{1,p}$, *provided* $1 < p < \infty$. *If* $p = 1$, *this subspace is dense in* $\overset{\circ}{H}_{1,1}$, *with respect to the relaxed topology of* $H_{1,1}$.

Proof. There is a standard approach to proving the density of locally Lipschitz functions in $H_{1,p}$, which has been applied previously, starting with various definitions of the space $H_{1,p}$; compare, e.g. [K]. This approach works in our context as well. For completeness, we indicate the argument.

Assume first that $1 < p < \infty$. It suffices to show that if $f \in H_{1,p}(U)$ and $B_{6d}(\underline{z}) \subset U$, then $f\,|\,B_d(\underline{z})$ is the limit in $H_{1,p}(B_d(\underline{z}))$, of a sequence of Lipschitz functions $\{f_n\}$.

By (4.19), for all $0 < n < \infty$, there exists $A_n \subset B_d(\underline{z})$, such that the Lipschitz constant of $f\,|\,A_n$ is bounded above by n, and (as follows from the weak type-(1,1) estimate) $\mu(B_d(\underline{z}) \setminus A_n) = o(n^{-1})$ as $n \to \infty$.

Let $f_n : B_d(\underline{z}) \to \mathbf{R}$ be the Lipschitz function, with $f_n\,|\,A_n = f\,|\,A_n$ and Lipschitz constant, $\mathbf{Lip}\, f_n \leq n$, obtained from MacShane's Lemma, (8.2) (with A of (8.2) $= A_n$). Clearly, for $z \in A_n$ a Lebesgue point, we have $\mathrm{Lip}\,(f_i - f_j)(z) = 0$, for all $i, j \geq n$. Thus, the estimate, $\mu(B_d(\underline{z}) \setminus A_n) = o(n^{-1})$ as $n \to \infty$, together with (4.3) and $f_i \,|\, A_j = f \,|\, A$, for $i, j \geq n$, implies that the sequence, $\{f_n\}$, is Cauchy in $H_{1,p}(B_d(\underline{z}))$. Since, $f_n \,|\, A_n = f \,|\, A$, it follows that $f_n \xrightarrow{H_{1,p}} f$.

We point out that the weak type (1,1)-estimate enables us to avoid an appeal to Mazur's lemma (valid only for $p > 1$); compare [K].

Given the density of locally Lipschitz functions in $H_{1,p}$, our assertion for $\overset{\circ}{H}_{1,p}$ follows by a straightforward argument in which a sequence of cut off functions is employed in a manner analogous to that in the proof of Lemma 2.17

If $p = 1$, the above argument can also be applied to find a sequence of locally Lipschitz functions converging in the $H_{1,1}$ norm to a given $f \in H_{1,1}$, provided we assume that f has a generalized upper gradient in L_1. This implies our assertions concerning the relaxed topology, for the case $p = 1$. □

Remark 4.25. If the set, U, satisfies a suitable chain condition as in [HKo], then it follows that the set of (globally) Lipschitz functions is dense in $H_{1,p}$.

Proposition 4.26. *Let (Z, μ) satisfy (0.1), and (4.3), for some $1 \leq p < \infty$. If f is Lipschitz and $\underline{z}$ is a Lebesgue point of g^p, for g a generalized upper gradient of f, then*

$$\mathrm{Lip}\, f(\underline{z}) \leq 2^{2\kappa+2} \tau g(\underline{z}) \,. \tag{4.27}$$

In particular, if $1 < p < \infty$ and $\underline{z}$ is a Lebesgue point of $(g_f)^p$, then

$$\mathrm{Lip}\, f(\underline{z}) \leq 2^{2\kappa+2} \tau g_f(\underline{z}) \,. \tag{4.28}$$

Proof. From assumption that $\underline{z}$ is a Lebesgue point of $(g_f)^p$, it follows that we can write $g_f = g_f(\underline{z}) + h$, where the normalized L_p-norm of the restriction of h to the ball, $B_r(\underline{z})$, goes to zero as $r \to 0$.

Fix $\eta > 0$. On $B_r(\underline{z})$ with its normalized measure, by Minkowski's inequality, we have $M_p(g_f) \leq g_f(\underline{z}) + M_p(h)$. Thus, the subset of $B_r(\underline{z})$ on which $M_p(h) > \eta$ contains the subset on which $M_p(g_f) > g_f(\underline{z}) + \eta$. By applying (4.16), (4.17), with h in place of g, it follows that on $B_r(\underline{z})$, the relative measure of the former subset (and hence, the latter) is as small as we like, provided that we choose r sufficiently small.

Fix $\epsilon > 0$. It follows from (4.19) and (0.1), that for $r \leq r(g_f, \epsilon)$ sufficiently small and all z with $\overline{\underline{z}, z} = r$, there exist z_1, z_2 with $\overline{z_1, \underline{z}} \leq \epsilon r$, $\overline{z_2, z} \leq \epsilon r$ and $|f(z_1) - f(z_2)| \leq 2^{2\kappa+2} \tau g(\underline{z}) r$. Hence, $|f(z) - f(\underline{z})| \leq$

$2^{2\kappa+2}\tau g(\underline{z})r + 2\cdot \mathbf{Lip}\, f\cdot \epsilon r$. Since ϵ is arbitrary, this suffices to complete the proof. □

From Proposition 4.26 together with Theorem 3.14, we obtain the following corollary (which could actually be proved directly, without recourse to Proposition 4.26, and hence to Theorem 4.14).

Corollary 4.29. *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *If* f *is Lipschitz and* $\underline{z}$ *is a Lebesgue point of* $(g_f)^p$, *then* $g_f(\underline{z}) = 0$ *implies* $\mathrm{Lip}\, f(\underline{z}) = 0$. *In particular, if* f *is asymptotically generalized linear at* $\underline{z}$, *then* $g_f(\underline{z}) = 0$, *if and only if lip* $f(\underline{z}) = 0$, *if and only if* $\mathrm{Lip}\, f(\underline{z}) = 0$.

Proof. This follows immediately from Theorem 3.14. □

Corollary 4.30. *If* (Z,μ) *satisfies* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Let* $\{f_i\}$ *be a sequence of Lipschitz functions, such that* $f_i \xrightarrow{H_{1,p}} f$. *Then for all* $\epsilon > 0$, *there exists* N_ϵ, *such that* $|\mathrm{Lip}\,(f_i - f_j)|_{L_p} < \epsilon$, *for* $i, j \geq N_\epsilon$. *If in particular,* f *is Lipschitz, then* $\mathrm{Lip}\,(f - f_i) \xrightarrow{L_p} 0$ *and so* $\mathrm{Lip}\, f_i \xrightarrow{L_p} \mathrm{Lip}\, f$.

We now come to the uniqueness statement for our generalization of Rademacher's theorem. It will suffice to assume $Z \neq \{\underline{z}\}$, for some $\underline{z}$. The proof depends on the following four lemmas.

For $\underline{z} \in Z$, we put $\rho_{\underline{z}}(z) = \overline{\underline{z}, z}$.

Note that if z is an isolated point of Z, then from (0.1), we get $\mu(z) > 0$, and since $Z \neq \{z\}$, we have $\mu(Z \setminus \{z\}) > 0$ as well. By applying (4.3) to the characteristic function of $\{z\}$, we conclude that no such isolated point exists.

Lemma 4.31. *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Let* $\mu(A) > 0$ *and let* $\underline{z}$ *be a Lebesgue point of* A. *Then there exists* $B \subset A$, *with* $\mu(B) > 0$, *such that* $g_{\rho_{\underline{z}}}(z) > 0$, *for all* $z \in B$.

Proof. Assume that there exists no such subset B. Since $\mathrm{Lip}\,\rho_{\underline{z}} \leq 1$, by applying (4.3) to $\rho_{\underline{z}} \,|\, B_r(\underline{z})$, and letting $r \to 0$, we get $|\rho_{\underline{z}} - (\rho_{\underline{z}})_{\underline{z},r}| = o(r)$.

By the discussion preceding the statement of the lemma, there exists a sequence of points, $z_i \to \underline{z}$, with $z_i \in A$, $z_i \neq \underline{z}$, for all i. From this together with (0.1), we easily contradict the assertion of the previous paragraph. □

Standard proofs of Rademacher's theorem for Lipschitz functions in $\mathbf{R}^n$ make use of the fact that the unit sphere has a countable dense subset; see for example, [EG]. A similar point occurs in the next lemmas.

Given real valued functions, $f_1, \ldots, f_k$, we put $f_{(a)} = a_1 f_1 + \cdots + a_k f_k$. A k-tuple, $(\tilde{a})$, will be called *rational*, if all its entries are rational numbers.

LEMMA 4.32. *Let (Z,μ) satisfy* (0.1), *and* (4.3), *for some* $1<p<\infty$. *For all* $1\le i\le k$, *let* $f_i : Z\to \mathbf{R}$ *be Lipschitz, with* $\mathrm{Lip}\, f_i\le L$. *Let* $Z_1\subset Z$ *denote the subset of points,* $\underline{z}$, *such that for all rational* k-*tuples,* $(\tilde a)$, *the function,* $f_{(\tilde a)}$, *is asymptotically generalized linear at* $\underline{z}$. *Then* $\mu(Z\setminus Z_1)=0$ *and the function,* $f_{(a)}$, *is asymptotically generalized linear at* $\underline{z}$, *for all* (a) *and all* $\underline{z}\in Z_1$. *Moreover,*

$$\left|g_{f_{(a')}}(\underline{z})-g_{f_{(a'')}}(\underline{z})\right|\le L\big(|a'_1-a''_1|+\cdots+|a'_k-a''_k|\big)\,, \tag{4.33}$$

for all $(a'),(a'')$ *and* $\underline{z}\in Z_1$.

Proof. Since the set of rational numbers is countable, it follows from Theorem 3.7, that $\mu(Z\setminus Z_1)=0$.

If $(a'_1,\dots,a'_k)$, and $(a''_1,\dots,a''_k)$, are arbitrary k-tuples, then by Proposition 1.11 and (3.3), for μ-a.e. z, we have

$$\left|g_{f_{(a')}}(z)-g_{f_{(a'')}}(z)\right|\le L\big(|a'_1-a''_1|+\cdots+|a'_k-a''_k|\big)\,. \tag{4.34}$$

This has the following consequences:

i) If z is a Lebesgue point of $(g_{f_{(a')}})^p$ and of $(g_{f_{(a'')}})^p$, then (4.34) holds.

ii) Let $\{(\tilde a)_j\}$, be a sequence of rational k-tuples converging to an arbitrary k-tuple (a). Since in particular, $\underline{z}\in Z_1$ is a Lebesgue point of $(g_{f_{(\tilde a)_j}})^p$, for all $f_{(\tilde a)_j}$, it follows easily that $\underline{z}$ is a Lebesgue point of $(g_{f_{(a)}})^p$ and $\lim_{j\to\infty} g_{f_{(\tilde a)_j}}(\underline{z})=g_{f_{(a)}}(\underline{z})$.

iii) Similarly, it follows that $f_{(a)}$ is asymptotically generalized linear at $\underline{z}\in Z_1$.

From ii), iii), it follows that $f_{(a)}$ is asymptotically generalized linear at $\underline{z}$, for all (a), and hence by i) that (4.33) holds for all (a) and all $\underline{z}\in Z_1$. This suffices to complete the proof. □

LEMMA 4.35. *Let (Z,μ) satisfy* (0.1), *and* (4.3), *for some* $1<p<\infty$. *Let* $f_1,\dots,f_k$, $g_{f_{(a)}}$, Z_1, *be as in Lemma 4.31. Let* $Z_2\subset Z_1$ *denote the subset of points,* $\underline{z}$, *such that* $\underline{z}$ *is a Lebesgue point of* $Lip\, f_{(\tilde a)}$, *for all rational* k-*tuples* $(\tilde a)$. *Then* $\mu(Z\setminus Z_2)=0$ *and* $\underline{z}$ *is a Lebesgue point of* $Lip f_{(a)}$, *for all* (a). *Moreover,*

$$g_{f_{(a)}}(\underline{z})\le Lip\, f_{(a)}(\underline{z})\le \mathrm{Lip}\, f_{(a)}(\underline{z})\le 2^{2\kappa+2}\tau g_{f_{(a)}}(\underline{z})\,, \tag{4.36}$$

for all (a) *and* $\underline{z}\in Z_2$.

Proof. By an argument completely analogous to that given in the proof of Lemma 4.32, it follows that $\mu(Z\setminus Z_2)=0$ and that $\underline{z}$ is a Lebesgue point of $Lip\, f_{(a)}$, for all (a) and $\underline{z}\in Z_2$. For all (a) we have $g_{f_{(a)}}(z)\le Lip\, f_{(a)}(z)$, for μ-a.e. z. Since $\underline{z}\in Z_2$ implies $\underline{z}$ is a Lebesgue point of $g_{f_{(a)}}$ and of

$Lip\, f_{(a)}$, it follows that $g_{f_{(a)}}(\underline{z}) \leq Lip\, f_{(a)}(\underline{z})$. In view Proposition 4.26, relation (4.36) holds as well. □

LEMMA 4.37. *Let (Z,μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Let f_i, $g_{f_{(a)}}$, $Z_1 \subset Z$, be as in Lemma 4.32. If for some $\underline{z} \in Z_1$ and all $(a) \neq 0$, we have $g_{f_{(a)}}(\underline{z}) > 0$, then $k \leq N(\kappa,\tau)$.*

Proof. Since $f_{(a)}$ is asymptotically generalized linear at $\underline{z}$, it follows from Theorem 3.14 that for all $\epsilon > 0$, there exists $r_\epsilon > 0$, such that the inequality, (4.9), holds (for a suitable constant) on $B_r(\underline{z})$, for all $0 < r < r_\epsilon$ and all $f_{(a)}$, for which the set, (a), lies in a certain ϵ-dense subset of the unit sphere in $\mathbf{R}^k$. By Lemma 4.32 and obvious continuity argument, this implies that (4.9) holds on $B_r(\underline{z})$, for all $0 < r < r_\epsilon$ and all $f_{(a)}$ (possibly with a slightly different constant). Thus, our assertion follows from Theorem 4.8. □

We emphasize that by means of Proposition 1.18, part iii) of the following Theorem 4.38 can immediately be rephrased in terms of the functions $f_{r,\underline{z}}$. In this way, we obtain the counterpart of (0.2)–(0.4) and in particular, the relative formulation of uniqueness given in section 0.

Given $f_1,\dots,f_k$ and f we put $f_{(a_0,(a))} = a_0 f + a_1 f_1 + \cdots + a_k f_k$. In i) below, the set, $Z_2(\alpha)$ denotes the subset defined in Lemma 4.35 with respect to the collection of Lipschitz functions, $f_1^\alpha,\dots,f_{k(\alpha)}^\alpha$. The set, $Z_2(f,\alpha)$ denotes the corresponding subset for the collection of Lipschitz functions, $f, f_1^\alpha,\dots,f_{k(\alpha)}^\alpha$.

We point out that the functions, $b_i^\alpha(z;f)$, which appear below, play the role of partial derivatives, $\partial f/\partial f_i^\alpha$. We also mention that although for the present, we are working with a fixed value of p, in actuality the sets, $U_\alpha, V_\alpha(f)$, of Theorem 4.38 below, can be chosen so that the conclusions hold, no matter which value, $p' \in [p,\infty)$, is used in defining the notion of asymptotic generalized linearity; see Corollary 6.38.

Theorem 4.38 (Relative uniqueness; Rademacher 2)**.** *There exists $N = N(\kappa,\tau)$ such that the following holds. Let (Z,μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Then there exists a countable collection of measurable sets, U_α, with $\mu(U_\alpha) > 0$, and Lipschitz functions, $f_1^\alpha,\dots,f_k^\alpha : Z \to \mathbf{R}$, with, $1 \leq k = k(\alpha) \leq N$, such that $\mu(Z \setminus \cup_\alpha U_\alpha) = 0$, and for all $\alpha, f_{(a)}$, the following holds:*

i) *$U_\alpha \subset Z_2(\alpha)$. In particular, for all $\underline{z} \in U_\alpha$, the function, $f_{(a)}^\alpha$, is asymptotically generalized linear at $\underline{z}$ and (4.33), (4.36) hold.*

ii) *$g_{f_{(a)}^\alpha}(\underline{z}) > 0$, for all $(a) \neq 0$ and $\underline{z} \in U_\alpha$.*

iii) *For* $f : Z \to \mathbf{R}$ *Lipschitz, there exists* $V_\alpha(f) \subset Z_2(f,\alpha) \cap U_\alpha$, *with* $\mu(U_\alpha \setminus V_\alpha(f)) = 0$, *and Borel functions,* $b_1^\alpha(z;f),\dots,b_k^\alpha(z;f)$, *of class* L_∞, *such that if* $\underline{z} \in V_\alpha(f)$, *then* $g_{f_{(-1,(a))}}(\underline{z}) = 0$, *if and only if* $Lip\, f_{(-1,(a))}(\underline{z}) = 0$, *if and only if* $\operatorname{Lip} f_{(-1,(a))}(\underline{z}) = 0$, *if and only if* $(a) = (b^\alpha)(\underline{z};f)$.

Proof. Let $\mu(A) > 0$. Clearly, it suffices to show that there exists, $U \subset A$, with $\mu(U) > 0$, satisfying i)–iii) above.

It follows from Lemmas 4.31, 4.37, that for any A, with $\mu(A) > 0$, there exists a maximal k, wth $1 \le k \le N(\kappa,\tau)$, such that i), ii) above hold on some subset of positive measure, $U \subset A$, for some collection of Lipschitz functions, $f_1,\dots,f_k$.

Given f, define the subset Z_2 as in Lemma 4.35, with respect to the functions $f, f_1,\dots,f_k$. Then $\mu(U \setminus Z_2) = 0$. By the maximality of k, there exist $V(f) \subset (Z_2 \cap U)$, such that $\mu(U \setminus V(f)) = 0$ and for $\underline{z} \in V(f)$, there exists $(b)(\underline{z})$, such that $g_{f_{(-1,(b)(\underline{z};f)}}(\underline{z}) = 0$.

By (4.36), at points of Z_2, we have $g_{f_{(-1,(a))}}(\underline{z}) = 0$, if and only if $Lip\, f_{(-1,(a))}(\underline{z}) = 0$, if and only if $\operatorname{Lip} f_{(-1,(a))}(\underline{z}) = 0$.

If for $\underline{z} \in V(f)$, we also have $g_{f_{(-1,(a))}}(\underline{z}){=}0$ (equivalently, $g_{(1,-(a))}(\underline{z}){=}0$) then by (3.3), we get $g_{f_{(b)(\underline{z};f)-(a)}}(\underline{z}) = 0$. This contradicts ii) unless $(a) = (b)(\underline{z})$. (Alternatively, in view of Proposition 1.16 and the previous paragraph, we could use $\operatorname{Lip} f_{(-1,(b)(\underline{z};f))}(\underline{z}) = 0$, $\operatorname{Lip} f_{(-1,(a))}(\underline{z}) = 0$ to obtain this contradiction.) This suffices to complete the proof of the existence and uniqueness of the $(b^\alpha)(\underline{z};f)$.

To see that these functions are Borel, we use the fact that $(a) = (b)^\alpha(\underline{z};f)$, if and only if $\operatorname{Lip} f_{(-1,(a))}(\underline{z}) = 0$, together with the fact that for (a) varying in some compact set, the Borel fuctions, $\operatorname{Lip} f_{(-1,(a))}(\underline{z})$, can be approximated in the sense of Lusin's theorem, (uniformly in (a)) by *continuous* functions. This follows from (1.16), together with

$$\big|\operatorname{Lip}{}_{f_{(-1,(a'))}}(\underline{z}) - \operatorname{Lip}{}_{f_{(-1,(a''))}}(\underline{z})\big| \le L\big(|a_1' - a_1''| + \cdots + |a_k'(\alpha) - a_k''(\alpha)|\big)\,. \tag{4.39}$$

For all $\underline{z} \in U_\alpha$, the function, $\operatorname{Lip} f_{(a)}(\underline{z})$, has a positive lower bound when (a) varies in the unit sphere in $\mathbf{R}^{k(\alpha)}$. It is easy to see that we can assume that the collection, $\{(U_\alpha, f_1^\alpha,\dots,f_{k(\alpha)}^\alpha)\}$, has been chosen such that this lower bound is uniform. From this and an obvious rescaling argument, it follows with (4.39), that the $(b^\alpha)(z;f)$ are of class L_∞. □

Remark 4.40. If in Theorem 4.38, we had been willing to replace $Lip\, f_{(a)}$ by $lip\, f_{(a)}$, then by appealing to Theorem 3.14, we could have omitted Lemma 4.35 and worked throughout with the set, Z_1, in place of Z_2.

Corollary 4.41. *If $\underline{z} \in V_\alpha(f)$, then $g_f(\underline{z}) = g_{f^\alpha_{(b)^\alpha(\underline{z};f)}}(\underline{z})$.*

Proof. For all (a) and μ-a.e. z, we have $g_f(z) \le g_{f-f^\alpha_{(a)}}(z) + g_{f^\alpha_{(a)}}(z)$ and $g_{f^\alpha_{(a)}}(z) \le g_f(z) + g_{-f+f^\alpha_{(a)}}(z)$. In particular, this holds for all $z = \underline{z} \in V_\alpha(f)$. If we take $(a) = (b^\alpha)(\underline{z}; f)$ and use $g_{f^\alpha_{b^\alpha(\underline{z};f)}}(\underline{z}) = g_{-f+f^\alpha_{b^\alpha(\underline{z};f)}}(\underline{z}) = 0$, the claim follows. □

For μ-a.e. $\underline{z} \in U_{\alpha_1} \cap U_{\alpha_2}$, the functions, $\{f_{1,i_{0,\underline{z}}}\}$, corresponding to U_{α_1}, can be expressed as linear combinations of the functions, $\{f_{2,j_{0,\underline{z}}}\}$, corresponding to U_{α_2} and vice versa. In particular, if $\mu(U_{\alpha_1} \cap U_{\alpha_2}) \not\equiv 0$, we have $k(\alpha_1) = k(\alpha_2)$. However, these values could differ if $\mu(U_{\alpha_1} \cap U_{\alpha_2}) = 0$.

In this way, we obtain μ-a.e. defined matrices of L_∞ functions, which satisfy a 1-cocycle relation on the common domain of any three of them. Hence, this collection determines a finite dimensional L_∞ vector bundle, T^*Z. By Theorem 4.38, a Lipschitz function, f, determines an L_∞ section of this bundle. Recall in this connection, that a section of an of L_∞ vector bundle is a collection of local sections, one for each U_α, which satisfy the appropriate compatibility condition, μ-a.e. on intersections, $U_{\alpha_1} \cap U_{\alpha_2}$.

Definition 4.42. The L_∞ section, df, of T^*Z determined by the Lipschitz function, f, is called the *differential* of f.

Note that the differential obviously satisfies the Leibnitz rule,

$$d(f_1 \cdot f_2) = df_1 \cdot f_2 + f_1 \cdot df_2\,. \tag{4.43}$$

Remark 4.44. Although an L_∞ vector bundle does not have topological invariants, there do exist topological invariants of operators between such bundles. For example, in the present context, one can define a kind first de Rham cohomology group, $H^1_{dR}(Z,\mu)$, of sections which are locally differentials of Lipschitz functions modulo those which are globally such. Clearly, this cohomology group is a bi-Lipschitz invariant of Z which depends only on the measure class of μ. In a similar spirit, one can define a first L_p-cohomology group.

Let (Z_1,μ_1) satisfy (0.1), and (4.3), for some $1 < p < \infty$, and let Z_2 be an arbitrary metric space. If $F : Z_1 \to Z_2$ and $f : Z_2 \to \mathbf{R}$ are Lipschitz, then $f \circ F$ is Lipschitz and $d(f \circ F)$ is defined μ_1-a.e. If in addition, (Z_2,μ_2) satisfies (0.1), and (4.3), for some $1 < p < \infty$, then df is defined as well. Moreover, given $U^1_\alpha \subset Z_1$, $U^2_\beta \subset Z_2$, as in Theorem 4.38, the corresponding transposed *Jacobian matrix* is defined (in obvious fashion) μ_1-a.e. on $F^{-1}(U^2_\beta) \cap U^1_\alpha$. However, in principle, we might have $\mu_1(Z_1 \setminus \cup_\beta F^{-1}(U^2_\beta)) > 0$, unless we further assume that $\mu_2(A) = 0$

implies $\mu_1(F^{-1}(A)) = 0$, or equivalently, that the *push forward measure*, $F_*(\mu_1)$, is *absolutely continuous* with respect to μ_2. In that case, there is a *natural induced map*, $F^* : T^*Z_2 \to T^*Z_1$, with the usual functorial properties. In particular, $F^*(df) = d(f \circ F)$ and $d(f \circ F)$ can be calculated in the usual fashion (i.e. from the transposed Jacobian matrix) μ_1-a.e. on $F^{-1}(U^2_\beta) \cap U^1_\alpha$.

REMARK 4.45. Consider $(Z_1, \mu_1), (Z_2, \mu_2)$, with μ_1, μ_2 Ahlfors Q-regular, for some $Q > 1$. Assume in addition that (4.3) holds for some $1 < p < Q$. According to [HeKo2], if $F : Z_1 \to Z_2$ is *quasiconformal* then $F_*(\mu_1)$, is absolutely continuous with respect to μ_2 (and $(F^{-1})_*(\mu_2)$, is absolutely continuous with respect to μ_1). Moreover, there exist sets, $A_{i,j}$, with $\mu_1(Z_1 \setminus \cup_{i,j} A_{i,j}) = 0$, such that the following holds.

i) Every point of $A_{i,j}$ is a Lebesgue point.

ii) $B_{(iK)^{-1}r}(F(\underline{z})) \subset F(B_r(\underline{z})) \subset B_{iKr}(F(\underline{z}))$, for $\underline{z} \in A_{i,j}$, $r \le j^{-1}$.

It follows that if $f : Z_2 \to \mathbf{R}$ is Lipschitz, then $f \circ F \,|\, A_{i,j}$ is Lipschitz as well. Moreover, if $\underline{z} \in A_{i,j}$ and $\tilde{f}$ is any Lipschitz extension of $f \,|\, A_{i,j}$, then $|f(z) - \tilde{f}(z)| = o(\overline{z, \underline{z}})$. As a consequence, f is asymptotically generalized linear μ_1-a.e. Hence, as above, there exists a natural induced map, $F^* : T^*Z_2 \to T^*Z_1$, such that $F^*(df) = d(f \circ F)$, and for which $d(f \circ F)$ can be calculated in the usual fashion, μ_1-a.e. on $F^{-1}(U^2_\beta) \cap U^1_\alpha$, from the transposed Jacobian matrix.

The properties of the function, $f \circ F$, can be abstracted. Thus, rather than considering Lipschitz functions on Z_2, we can consider from the start, more general functions, f, for which there exists a decomposition of the space, (Z_2, μ_2), whose properties, with respect to f, are analogous to those of the decomposition, $\{A_{i,j}\}$, for $f \circ F$. As above, this class of functions is stable under quasiconformal maps, given our previous assumptions (i.e. $1 < p < Q$, etc.). In this way, our discussion can be made more symmetrical.

REMARK 4.46. The discussion of Remark 4.45 and of the paragraph which preceded it should be compared to that of section 14. There we consider induced maps on tangent cones and their adjoints.

Theorem 4.47 (Existence and uniqueness of strong derivatives)**.** *Let (Z, μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Then $f \in H_{1,p}$ if and only if there exists a sequence of Lipschitz functions, $f_i \xrightarrow{L_p} f$, such that $df \xrightarrow{L_p} v$, for some v. Moreover, if $f \in H_{1,p}$, then v is unique.*

Proof. Since for Lipschitz functions, we have $|df(\underline{z})| = g_f(\underline{z})$, for μ-a.e. $z \in Z$ (see Corollary 4.41) this follows from Theorem 4.24, together with the fact that the map $i : H_{1,p} \to L_p$ is an injection; see Remark 2.4 and Theorem 2.7. □

It follows from Corollary 4.41, that for all $\underline{z} \in U_\alpha$, we can define a norm, $|(a)|_{\alpha,\underline{z}}$, on the space, $\mathbf{R}^{k(\alpha)}$, such that for f Lipschitz, and $\underline{z} \in V_\alpha(f)$, we have $|(b^\alpha)(\underline{z}; f)|_{\alpha,\underline{z}} = g_f(\underline{z})$. We will also refer to this norm as *the norm on* $T^*Z_{\underline{z}}$.

Recall that any norm, $|\cdot|$, on a k-dimensional vector space, $\mathcal{V}$, is $c(k)$-quasi-isometric to a distinguished inner product norm, $||\cdot||$, and in particular, to a *uniformly* convex norm. The corresponding inner product on the dual space, $\mathcal{V}^*$, is gotten by identifying the functions of $\mathcal{V}^*$ with their restrictions to the unit ball, $B_1(0)$ (as defined by $|\cdot|$) and regarding the functions so obtained elements of $L_2(B_1(0))$, where the measure is $\underline{c}(k)\cdot\mathcal{H}^k$. Here $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure associated to the metric defined by $|\cdot|$, and $\underline{c}(k) = (k+1)V(k)/V(k+2)$, where $V(n)$ denotes the volume of the unit ball in $\mathbf{R}^n$. If the norm, $|\cdot|$, happens to come from an inner product, then we have $||\cdot|| = |\cdot|$. In general, it is a standard fact (and not difficult to verify) that $|\cdot|$, and $||\cdot||$ are $c(k)$-quasi-isometric.

From Theorem 4.47 and the discussion of the previous paragraph, it follows that the space, $H_{1,p}$, has a canonical uniformly convex norm, $||\cdot||_{1,p}$, which is $c(\kappa,\tau)$-quasi-isometric to its usual norm $|\cdot|_{1,p}$. The norm, $||\cdot||_{1,p}$, is the global norm obtained by integration with respect to μ of the pointwise norm obtained by replacing each norm, $|(a)|_{\alpha,\underline{z}}$, on the space, $\mathbf{R}^{k(\alpha)}$, by the canonically associated inner product norm, as described in the preceding paragraph.

Thus, we get the following very important conclusion, which, via Theorem 4.53 below, plays a crucial role in the proof of Theorems 5.1, 6.1, 12.7.

Theorem 4.48. *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Then the norm on* $H_{1,p}$ *is equivalent to a uniformly convex norm. In particular, the space,* $H_{1,p}$ *is reflexive.*

Remark 4.49. If the pointwise norm on the space of differentials is *strictly* convex, then so is the global norm, $|\cdot|_{1,p}$, for $1 < p < \infty$, the conditions, $f \in H_{1,p}$, $f_i \xrightarrow{L_p} f$, $|g_{f_i}|_{L_p} \to |g_f|_{L_p}$, imply $f_i \xrightarrow{H_{1,p}} f$.

From Theorem 4.48 and Mazur's lemma, we immediately get:

Theorem 4.50. *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *If* $\{f_i\}$ *is a bounded sequence in* $H_{1,p}$ *such that* $f_i \xrightarrow{L_p} f$, *then there exists a sequence,* $\hat{f}_j \xrightarrow{H_{1,p}} f$, *such that each function,* $\hat{f}_j$ *is a finite convex* *combination*, $\hat{f}_j = \sum_0^{N_j} a_{i,j} f_i$.

As a corollary of Theorem 4.50 we have:

Theorem 4.51. *Let (Z,μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Let $||\cdot||_{1,p}$ be a norm on $H_{1,p}$ equivalent to $|\cdot|_{1,p}$. If $f_i \xrightarrow{L_p} f$, then,*

$$\liminf_i ||f_i||_{1,p} \geq ||f||_{1,p} \,. \tag{4.52}$$

In particular, this holds for the norm, $||f||_{1,p} = |f|_{L_p} + |\mathrm{Lip}\, f|_{L_p}$.

We call $|\mathrm{Lip}\, f|^p_{L_p}$, the *Dirichlet p-energy*.

A *norm*, $||\cdot||$, on T^*Z, is an assignment to every measurable section, ψ, of T^*Z, a measurable function $||\psi(z)||$, defined over $\cup_\alpha U_\alpha$, such that for all α and $z \in U_\alpha$, the function on $\mathbf{R}^{k(\alpha)}$, defined by $||(a)||_{\alpha,z} = ||df^\alpha_{(a)}(z)||$, is a norm on $\mathbf{R}^{k(\alpha)}$. We say that $||\cdot||$ is an *equivalent norm* on T^*Z, if there exists $0 < c < \infty$, such that $\lambda^{-1}|\psi(z)| \leq ||\psi(z)|| \leq \lambda|\psi(z)|$.

For example, given (0.1), (4.3), the norm for which $||(a)||_{\alpha,z} = \mathrm{Lip}\, f^\alpha_{(a)}(z)$ (i.e. $||df(z)|| = \mathrm{Lip}\, f(z)$) is an equivalent norm on T^*Z; see Proposition 4.26 and compare Corollary 4.30.

The following theorem, will have several very significant applications; see in particular, Theorems 5.1, 6.1, 12.5.

Theorem 4.53. *Let (Z,μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Let $||\cdot||$ be an equivalent norm on T^*Z. If $\{f_i\}, \{h_i\}$, are sequences, with h_i bounded in L_p, such that*

$$f_i \xrightarrow{L_p} f \,, \tag{4.54}$$

$$\big||df_i(z)|\big| \leq h_i(z) \qquad (\text{for } \mu\text{-a.e. } z) \,, \tag{4.55}$$

then

$$\big||df(\underline{z})|\big| \leq \limsup_{i\to\infty} h_i(z) \qquad (\text{for } \mu\text{-a.e. } \underline{z}) \,. \tag{4.56}$$

Proof. Let $\hat{f}_j = \sum_{i=0}^{N_j} a_{i,j} f_i$ be as in Theorem 4.50. Put $\hat{h}_j = \sum_{i=0}^{N_j} a_{i,j} h_i(z)$. Then for μ-a.e. $\underline{z}$,

$$\big||d\hat{f}_j(\underline{z})|\big| \leq \sum_{i=0}^{N_j} a_{i,j} ||df_j(\underline{z})|| \leq \hat{h}_j(\underline{z}) \,. \tag{4.57}$$

We have $\limsup_{j\to\infty} \hat{h}_j \leq \limsup_{i\to\infty} h_i$, and since $\hat{f}_j \xrightarrow{H_{1,p}} f$, we also get $||d\hat{f}_j|| \xrightarrow{L_p} ||df||$. This, together with (4.57), suffices to complete the proof. □

Define a *riemannian metric* on (Z,μ), to be an equivalent norm on T^*Z, such that for all $\alpha, z \in U_\alpha$, the norm, $||\cdot||_{\alpha,z}$, is given by an inner product. It follows from the discussion preceding Theorem 4.48, that if (Z,μ) satisfies (0.1), (4.3), then Z carries a canonical riemannian metric.

As above, a riemannian metric determines a norm on $H_{1,p}$ which is equivalent to the usual one. In particular, in the presence of the type (1,2) Poincaré inequality,

$$\left(|f - f_{\underline{z},r}|^2\right)_{\underline{z},r} \leq \tau^2 s^2 (g^2)_{\underline{z},r}\,, \tag{4.58}$$

the space, $H_{1,2}$ carries a natural Dirichlet form.

Theorem 4.59. *Let (Z,μ) satisfy (0.1), (4.58). Then for every L_∞ riemannian metric, the associated Dirichlet form is closable and hence, determines a canonical self-adjoint operator, Δ on $L_2(Z)$. If in addition, Z is compact, then $(1+\Delta)^{-1}$ is a compact operator.*

Proof. Since $i : H_{1,2} \to L_2$ is an injection or (what is essentially equivalent in our context) since strong derivatives are unique in the sense of Theorem 4.47, it follows by a standard argument that the Dirichlet form is closable and hence, defines a unique self-adjoint operator; see Theorems 2.5, 2.7 and [Fuk]. The compactness of $(1+\Delta)^{-1}$, given the compactness of Z, is then a standard consequence of Theorem 4.8. □

Remark 4.60. In view of (4.43), one can derive Caccioppoli type inequalities for eigenfunctions and their reciprocals; compare [HeKM]. Also, by Theorem 4.47 and the results of [HKo] which were recalled in section 2 (see (4.4), (4.5), the four conditions in the definition of "p-admissible weight" hold; see [HeKM]. In view of these considerations the Hölder continuity of the eigenfunctions can be proved by arguing as in the proof of Theorem 6.6 of [HeKM], which is based in part on Moser iteration. (Although the context considered in [HeKM] is that of measures on $\mathbf{R}^n$ defined by weight functions, the essential point is the validity of the above mentioned four properties of the measure.)

Remark 4.61. In [St], self-adjoint Laplace operators on metric measure spaces are obtained in a more restricted context, by an entirely different method.

Example 4.62. With its standard Carnot-Caratheodory metric, the 3-sphere, S^3, has topological dimension 3 and Hausdorff dimension 4; see [Gro2]. The cotangent bundle in our sense is just the dual of the horizontal distribution, and hence, has dimension 2. For f Lipschitz, then g_f, is the norm of the restriction of *df* to the horizontal distribution and the associated Laplacian is the standard *linear* self-adjoint subelliptic Laplacian in this case.

Let $\mathcal{H}^k$ denote k-dimensional Hausdorff measure and let dim denote Hausdorff dimension.

In Theorem 13.4, it will be shown that for μ-a.e. $\underline{z} \in U_\alpha$, and all tangent cones, $Z_{\underline{z}}$, there exists a surjective Lipschitz map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$, where, by definition, $TZ_{\underline{z}} = (T^*Z_{\underline{z}})^*$. In particular, $\mathcal{H}^{k(\alpha)}(B_r(w)) > c(k(\alpha))r^{k(\alpha)}$, for all $B_r(w) \subset Z_{\underline{z}}$.

Let $f^\alpha : U_\alpha \to \mathbf{R}^{k(\alpha)}$ be defined by $f^\alpha(z) = (f_1^\alpha(z), \dots, f_{k(\alpha)}^\alpha(z))$.

Conjecture 4.63. $\mathcal{H}^{k(\alpha)}(f^\alpha(U_\alpha)) > 0$.

Conjecture 4.63 holds in the very special case in which the doubling constant, κ, satisfies $\kappa = k(\alpha)$; see Theorem 13.12.

Remark 4.64. One can ask moreover, if U_α has a sort of measurable generalized product structure, corresponding (roughly) to a *rectifiable* direction of dimension $k(\alpha)$ and a transverse direction in which the differentials of all Lipschitz functions vanish. Such a formulation should account for examples such as the Carnot-Caratheodory spaces of [Gro2], as well as the spaces of nonintegral Hausdorff dimension of [BoP] and [L].

Motivation for our next conjecture is provided by the discussion of section 11; compare also section 14.

Conjecture 4.65. *If Z is a length space and on each set U_α, we have $k(\alpha) = \dim U_\alpha$ then for μ-a.e. $\underline{z}$, all tangent cones are Minkowski spaces, i.e. finite dimensional normed linear spaces. Moreover, in this case, the space, Z, is μ-rectifiable; see [F].*

Remark 4.66. By employing considerations which are different from ours, N. Weaver has independently defined a notion of cotangent bundle for any metric measure space. In general, his cotangent bundle need not be finite dimensional. However, given (0.1), and (4.3), for some $1 < p < \infty$, his cotangent bundle and ours are naturally isomorphic. The proof of this assertion, which was worked out jointly with Weaver, relies on the results of the present section, in particular, the finite dimensionality of our cotangent bundle and the concommitant the reflexivity of $H_{1,p}$, for $1 < p < \infty$; for details see [W]. Weaver has verified that for Carnot-Carathéodory and Laakso spaces, his cotangent bundle (and hence, ours) has the dimension the expected dimension, i.e. the dimension of the "horizontal" subspace of any tangent cone; see [W].

5 The Length Space Condition and $g_f = \operatorname{Lip} f$

The main result of this section is:

Theorem 5.1. *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$, *with* Z *a complete length space. If* f *is locally Lipschitz, then* $g_f(\underline{z}) = \operatorname{Lip} f(\underline{z})$, *for* μ*-a.e.* $\underline{z}$.

In Theorem 6.1, we will remove the assumption that Z is a length space, by showing that $\operatorname{Lip}_\rho f(\underline{z}) = \operatorname{Lip}_{\rho_0} f(\underline{z})$, for μ-a.e. $\underline{z}$. Here, $\operatorname{Lip}_\rho f(z)$ denotes the pointwise Lipschitz constant with respect to the underlying metric, ρ, and $\operatorname{Lip}_{\rho_0} f(z)$ denotes the pointwise Lipschitz constant with respect to the canonically associated length space metric, ρ_0; for the existence of ρ_0, see section 17, [DSe2], and [Se5, Lemma 2.38].

Since our considerations are essentially local, we need only assume that Z is *locally a complete length space*, i.e. for all $z \in Z$, there exists $r(z) > 0$, such that for all $z_1, z_2 \in B_r(z)$, there is a geodesic, γ, from z_1 to z_2, with length, $\ell = \overline{z_1, z_2}$, the distance from z_1 to z_2. With the above understanding, *throughout the remainder of the paper*, for brevity, the adjectives "locally" and "complete" will often be omitted.

Recall that if Z is a length space satisfying no additional assumptions, and g is a *continuous* upper gradient for the Lipschitz function f, then it is obvious that $\operatorname{Lip} f(\underline{z}) \leq g(\underline{z})$, for *all* $\underline{z}$. By using this fact, together with Theorem 4.50, we can reduce the proof of Theorem 5.1 to the following result concerning upper gradients.

Let $\overline{B}_d(\overline{z})$ denote the *closed* ball of radius d. The reason for stipulating closed balls in Lemma 5.2 stems from the role of compactness in the proof of Lemma 5.18. (Recall that the existence of nontrivial measure satisfying (0.1) implies that balls of finite radius are totally bounded. Hence, if Z is complete, closed balls of finite radius are compact.)

Lemma 5.2. *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$, *with* Z *a length space. Let* $f \in L_p(\overline{B}_{12d}(\overline{z}))$, *and let* $g \in L_p(\overline{B}_{12d}(\overline{z}))$ *be a positive countably valued, lower semicontinuous function, with values in* $[\eta, \infty]$, *for some* $\eta > 0$. *If* g *is an upper gradient for* f, *then on* $B_d(\underline{z})$, *there exist sequences of Lipschitz functions,* $f_j \xrightarrow{L_p} f$ *and continuous upper gradients,* v_j *of* f_j, *such that* $\limsup_{j\to\infty} v_j \leq g$, μ*-a.e.*

Proof of Theorem 5.1. Since the statement is local, it suffices to consider fuctions defined on balls, $\overline{B}_{12d}(\overline{z})$ of finite nonzero radius.

Let $f_i \xrightarrow{L_p} f$, $g_i \xrightarrow{L_p} g_f$, with g_i an upper gradient for f_i. The Vitali-Caratheodory theorem asserts that for all $\lambda > 0$ there exists a countably valued, lower semicontinuous function, $g_{\lambda,i}$, such that $g_{\lambda,i} \geq \min(g_i, \eta_\lambda)$, for some $\eta_\lambda > 0$, and $|g_i - g_{\lambda,i}|_{L_p} \leq \lambda$. By choosing a sequence, $\lambda_i \to 0$, and replacing g_i by $g_{\lambda,i}$, we can assume without loss of generality that the functions g_i themselves, have the above mentioned properties.

We can apply Lemma 5.2 to obtain diagonal sequences, $\{f_{k,j(k)}\}$,$\{v_{k,j(k)}\}$, with $v_{k,j(k)}$ a continuous upper gradient for $f_{k,j(k)}$, such that $f_{k,j(k)} \xrightarrow{L_p} f$, $\limsup_{k\to\infty} v_{k,j(k)} \leq g_f$, μ-a.e. From the continuity of $v_{k,j(k)}$ we get $\mathrm{Lip}\, f_{k,j(k)} \leq v_{k,j(k)}$. By applying Theorem 4.53 (with $||df(z)|| = \mathrm{Lip}\, f(z)$) we get $\mathrm{Lip}\, f(\underline{z}) \leq g_f(\underline{z})$, for μ-a.e. $\underline{z}$. Since g_f is the minimal upper gradient, this suffices to complete the proof. □

By referring to the proof of Theorem 4.53, we see that in proving Theorem 5.1, the following strengthening of Theorem 4.24 has been established.

Theorem 5.3. *Let (Z,μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$, with Z a length space. For $1 < p < \infty$, there exists, $\hat{f}_{k,j(k)} \xrightarrow{H_{1,p}} f$, $\hat{v}_{k,j(k)} \xrightarrow{L_p} g_f$, where $\hat{v}_{k,j(k)}$ is a continuous upper gradient for $\hat{f}_{k,j(k)}$.*

The proof of Lemma 5.2 utilizes an auxilliary function, $F_g(z_1, z_2)$, associated to the function g. The initial part of the discussion of the properties of the function, F_g, requires no special assumptions on (Z,μ).

Fix $A \subset Z$. Given a nonnegative Borel function, $g : A \to [0,\infty]$, define an associated function, $F_g : A \times A \to [0,\infty]$, by

$$F_g(z_1, z_2) = \inf_c \int_0^\ell g(c(s))\, ds\,, \tag{5.4}$$

where the infimum is over all rectifiable curves, $c \subset A$, from z_1 to z_2. If no such curves exist, we put $F_g(z_1, z_2) = \infty$.

It is trivial to check that F_g satisfies,

$$F_g(z_1, z_2) = F_g(z_2, z_1)\,. \tag{5.5}$$

$$F_g(z, z) = 0\,. \tag{5.6}$$

$$F_g(z_1, z_2) + F_g(z_2, z_3) \geq F_g(z_1, z_3)\,. \tag{5.7}$$

Thus, F_g defines a length space pseudometric, which is obtained from the length space pseudometric canonically associated to the given metric, by a conformal deformation with conformal factor g.

Let $F_g(\underline{z}, z_1) < \infty$, $F_g(\underline{z}, z_2) < \infty$. From (5.5), (5.7), it follows that for this fixed $\underline{z}$, and all rectifiable curves, c, from z_1 to z_2, we have

$$\bigl|F_g(\underline{z}, z_1) - F_g(\underline{z}, z_2)\bigr| \leq \int_0^\ell g(c(s))\, ds\,. \tag{5.8}$$

Note however, that $F_g(\underline{z}, z_1) = \infty$, $F_g(\underline{z}, z_2) = \infty$, does *not* imply $F_g(z_1, z_2) = \infty$, Thus, if for some fixed $\underline{z}$, the function, $F_g(\underline{z}, z)$, is *not bounded*, then g, *need not* be an upper gradient for $F_g(\underline{z}, z)$; compare the sentence following (1.2). Below, we will obtain additional control over the function, F_g, given that (0.1), (4.3) hold; see Lemma 5.27.

Clearly, we have

$$F_{g_1} + F_{g_2} \le F_{g_1+g_2} \,. \tag{5.9}$$

Let $\rho(z_1, z_2) = \overline{z_1, z_2}$. If there exists a minimal geodesic from z_1 to z_2, then for any constant function, $\delta \ge 0$,

$$F_\delta(z_1, z_2) = \delta\rho(z_1, z_2)\,, \tag{5.10}$$

and so,

$$F_g(z_1, z_2) + \delta\rho(z_1, z_2) \le F_{g+\delta}(z_1, z_2)\,. \tag{5.11}$$

The following proposition is trivial to check.

PROPOSITION 5.12. *Let $g : A \to [0, \infty]$.*

i) *If g is an upper gradient for f, then for all z_1, z_2,*

$$\big|f(z_1) - f(z_2)\big| \le F_g(z_1, z_2)\,. \tag{5.13}$$

ii) *If for some fixed $\underline{z}$, the function, $F_g(\underline{z}, z)$ is bounded, then $g(z)$, is an upper gradient for $F_g(\underline{z}, z)$. In particular, if A is rectifiably connected and g is bounded, then $F_g(\underline{z}, z)$ is Lipschitz.*

Let $g : A \to [0, \infty]$. Let $f : E \to \mathbf{R}$ be *bounded*, where $E \subset A$, and assume that (5.13) holds for all $z_1, z_2 \in E$. Define the function, $f^{*,g} : A \to \mathbf{R}$, by

$$f^{*,g}(z) = \inf_{\underline{z}\in E} f(\underline{z}) + F_g(\underline{z}, z)\,, \tag{5.14}$$

and the function, $f_{*,g} : A \to \mathbf{R}$, by

$$f_{*,g}(z) = \sup_{\underline{z}\in E} f(\underline{z}) - F_g(\underline{z}, z)\,. \tag{5.15}$$

If we take $E = \{\underline{z}\}$, then the the following lemma reduces to (5.13); compare also (8.2), (8.3). The proof, which is a routine exercise, will be omitted.

LEMMA 5.16. *If (5.13) holds for all $z_1, z_2 \in E$, then the functions, $f_{*,g}, f^{*,g}$, have the following properties:*

i) $f_{*,g}\,|\,E = f\,|\,E$, $\quad f^{*,g}\,|\,E = f\,|\,E$.

ii) *If $\tilde f$ is any function such that $\tilde f\,|\,E = f\,|\,E$ and g is an upper gradient for $\tilde f$, then*

$$f_{*,g} \le \tilde f \le f^{*,g}\,. \tag{5.17}$$

iii) *If one of the functions* $f_{*,g}, f^{*,g}$, *is bounded then both are bounded and in this case,* $g(z)$ *is an upper gradient for* $f_{*,g}, f^{*,g}$. *In particular, if* A *is rectifiably connected and* g *is bounded, then the functions,* $f_{*,g}, f^{*,g}$, *are Lipschitz.*

In the following Lemma 5.18, we allow the possibility that the functions, g, F_g, take the value, ∞, on a set of positive measure.

LEMMA 5.18. *Let* A *be compact. Let* $g \in L_p$ *be a countably valued, lower semicontinuous function with values in* $[\eta, \infty]$, *for some* $\eta > 0$. *Then there exists a nondecreasing sequence of Lipschitz functions,* $\{u_j\}$, *such that* $\{u_j\}$ *converges pointwise to* g *and* $\{F_{u_j}\}$ *converges pointwise to* F_g.

Proof. The function, g, has the representation,

$$g = \eta + \sum_{k=1}^{\infty} e_k \chi_{\mathcal{U}_k}\,, \tag{5.19}$$

where $\mathcal{U}_k$ is a relatively open subset of A and $\chi_{\mathcal{U}_k}$ denotes the characteristic function of $\mathcal{U}_k$. Moreover, $\eta > 0$ and $e_k \geq 0$ for all k.

As a standard consequence, it follows that g is the pointwise limit of a nondecreasing sequence of Lipschitz functions. This can be seen as follows.

Write $\mathcal{U}_k = \cup_i K_{k,i}$, where for all k, i, the set $K_{k,i}$ is a compact subset of the interior of $K_{k,i+1}$. Let $\psi_{k,i} : Z \to [0,1]$ be a nonnegative Lipschitz function supported in $K_{k,i+1}$ such that $\psi_{k,i} \,|\, K_{k,i} \equiv 1$. Thus, if we put

$$u_j = \eta + \sum_{k=1}^{j} e_k \psi_{k,j}\,, \tag{5.20}$$

then $\{u_j\}$ is an nondecreasing sequence of Lipschitz functions whose pointwise limit is g.

Using the fact that the sequence, $\{u_j\}$, is nondecreasing, we now show that for all (z_1, z_2), we have

$$\lim_{j\to\infty} F_{u_j}(z_1, z_2) = F_g(z_1, z_2)\,. \tag{5.21}$$

Note first that it suffices to assume that there exists $L < \infty$, such that $\lim_{j\to\infty} F_{u_j}(z_1, z_2) = L$. Otherwise, there is nothing to prove. Also, we can assume that there exists at least one rectifiable curve from z_1 to z_2. Otherwise, $F_{u_j}(z_1, z_2) = F_g(z_1, z_2) = \infty$.

Since $u_j \geq \eta > 0$ is continuous and A is compact, it follows by an obvious compactness argument, that there exists $\ell < \infty$, such that for all j, there exists $c_j : [0, \ell_j] \to A$, with $\ell_j \leq \ell$, such that

$$F_{u_j}(z_1, z_2) = \int_0^{\ell_j} u_j(c_j(s))\, ds\,. \tag{5.22}$$

Similarly, there exists $c_\infty : [0, \ell_\infty] \to A$, such that after passing to a subsequence, we can assume that $\ell_j \to \ell_\infty$ and $c_j \to c_\infty$.

By the monotone convergence theorem, for all $\psi > 0$, there exists N_1, such that for $j \geq N_1$,

$$F_g(z_1, z_2) \leq \int_0^{\ell_\infty} g(c_\infty(s))ds \leq \int_0^{\ell_\infty} u_{N_1}(c_\infty(s))ds + \psi\,. \tag{5.23}$$

From the continuity of u_{N_1}, it follows that there exists N_2 such that for $j \geq N_2$, we have

$$\int_0^{\ell_\infty} u_{N_1}(c_\infty(s))\,ds \leq \int_0^{\ell_j} u_{N_1}(c_j(s))\,ds + \psi\,. \tag{5.24}$$

Since the sequence, $\{u_j\}$, is *nondecreasing*, for all $j \geq \max(N_1, N_2)$, we get

$$\int_0^{\ell_j} u_{N_1}(c_\infty(s))\,ds \leq \int_0^{\ell_j} u_j(c_j(s))\,ds + \psi = F_{u_j}(z_1, z_2) + \psi\,. \tag{5.25}$$

Relations (5.23)–(5.25), give (5.21). □

REMARK 5.26. As pointed out to us by Seppo Rickman, there is a strong similarity between the proof of Lemma 5.18 and the proof of Ziemer's theorem concerning the equality of the capacity and the modulus for condensers; compare also Theorem 9.6 and see [Ri, p. 54].

We now show that if (0.1), (4.3) hold, then we can obtain additional control over the function $F_g(\underline{z}, z)$.

As in section 4, for a given function, h, let $M_p h$ denote the maximal function of h; see (4.22), (4.23). Note that $M_p(c + h) \leq |c| + M_p h$, for any constant function c.

LEMMA 5.27. *Let (Z, μ) satisfy (0.1), and (4.3), for some $1 \leq p < \infty$, with Z a length space. Let $g : U \to [0, \infty]$, $g \in L_p(U)$. If $M_p g(\underline{z}) = t < \infty$ and $B_r(\underline{z}) \subset U$ then:*

i) *$g \,|\, B_r(\underline{z})$ is a generalized upper gradient for $F_g(\underline{z}, z) \,|\, B_r(\underline{z})$. Moreover,*

$$⨍_{B_r(\underline{z})} F_g(\underline{z}, z)\,d\mu \leq 2^{4\kappa}\tau r t\,, \tag{5.28}$$

and

$$\left(⨍_{B_r(\underline{z})} (F_g(\underline{z}, z))^p\,d\mu \right)^{1/p} \leq (2^{4\kappa} + 1)\tau r t\,, \tag{5.29}$$

ii) *In particular,* $F_g(\underline{z}, z) < \infty$ *for* μ*-a.e.* $z \in B_r(\underline{z})$.

iii) *If* $M_p g(\underline{z}_k) = t < \infty$, $B_{r_k}(\underline{z}_k) \subset U$, *for* $k = 1, 2$, *and* $B_{r_1}(\underline{z}_1) \cap B_{r_2}(\underline{z}_2) \neq \emptyset$, *then* $F_g(\underline{z}_1, \underline{z}_2) < \infty$.

Proof. By the Vitali-Caratheodory theorem, we can assume that g is countably valued, lower semicontinuous, with values in $[\eta, \infty]$, for some $\eta > 0$.

Let $\{u_j\}$ be as in by Lemma 5.18. By the monotone convergence theorem, we have

$$⨍_{B_r(\underline{z})} F_g(\underline{z}, z)\, d\mu = \lim_{j\to\infty} ⨍_{B_r(\underline{z})} F_{u_j}(\underline{z}, z)\, d\mu\,. \tag{5.30}$$

Since the function, $F_{u_j}(\underline{z}, z)$ is Lipschitz, every point (and in particular, $\underline{z}$) is a Lebesgue point of this function. Thus, the right-hand side of (5.30) can be bounded by the "telescope" estimate (4.22). This gives (5.28), which directly implies ii), iii).

Finally, from (5.30), (4.3) and the monotone convergence theorem, it follows easily that $F_{u_j}(\underline{z}, z) \xrightarrow{L_p} F_g(\underline{z}, z)$, on $B_r(\underline{z})$ and that (5.29) holds. Thus, $g \,|\, B_r(\underline{z})$ is a generalized upper gradient for $F_g(\underline{z}, z) \,|\, B_r(\underline{z})$, as asserted. □

Proof of Lemma 5.2. For $0 \le t < \infty$, let $C_t \subset B_d(\overline{z})$ denote the subset consisting of Lebesgue points, z, of f such that $M_p g(z) < t$. Since $g \in L_p$, the weak type (1,1) estimate implies that $\cup_t C_t$ has full measure in $B_d(\overline{z})$. Moreover, $f \,|\, C_t$ is Lipschitz, with Lipschitz constant $L(\kappa, \tau, t, d)$; see (4.19).

Fix $\epsilon > 0$. Let $G_{\epsilon,t} = \{z_{\epsilon,t,k}\}$ denote a maximal subset of C_t, such that distinct points of $G_{\epsilon,t}$ are at mutual distance at least ϵ. Thus, $G_{\epsilon,t}$ is ϵ-dense in C_t. Moreover, by a standard result, (0.1), implies that there is a definite bound on the cardinality of $G_{\epsilon,t}$. Clearly, we can arrange

$$G_{\epsilon,t} \subset G_{\epsilon',t'} \qquad (\text{if } \epsilon' \le \epsilon,\ t \le t')\,. \tag{5.31}$$

By Lemma 5.27, $F_g(\underline{z}_1, \underline{z}_2) < \infty$ for $\underline{z}_1, \underline{z}_2 \in C_t$. Since, $G_{\epsilon,t}$ is a finite set, it follows that for all $\delta > 0$, there exists $N(\epsilon, \delta, t) < \infty$, such that for $j \ge N(\epsilon, \delta, t)$, and $z_{\epsilon,k_1}, z_{\epsilon,k_2} \in G_{\epsilon,t}$,

$$0 \le F_g(z_{\epsilon,k_1}, z_{\epsilon,k_2}) - F_{u_j}(z_{\epsilon,k_1}, z_{\epsilon,k_2}) < \tfrac{1}{2}\epsilon\delta\,. \tag{5.32}$$

Thus, by (5.9), (5.10), (5.13), we get

$$\begin{aligned} \big|f(\underline{z}_{k_1,t}) - f(\underline{z}_{k_2,t})\big| &\le F_g(\underline{z}_{k_1,t}, \underline{z}_{k_2,t}) \\ &\le F_{u_j+\delta}(\underline{z}_{k_1,t}, \underline{z}_{k_2,t}) \quad (\text{if } j \ge N(\epsilon, \delta, t)). \end{aligned} \tag{5.33}$$

Let $j \ge N(\epsilon, \delta, t)$. In view of (5.33), we can apply Lemma 5.16, with $B = G_{\epsilon,t}$ and g of that lemma replaced by $u_j + \delta$. Let $f_{\epsilon,\delta,t,j} = f_{*,u_j+\delta}$. We also put $f_{\epsilon,\delta,t,\infty} = f_{*,g+\delta}$, $f^{\epsilon,\delta,t,\infty} = f^{*,g+\delta}$. Thus,

$$f_{\epsilon,\delta,t,\infty} \le f_{\epsilon,\delta,t,j} \le f^{\epsilon,\delta,t,\infty}\,, \tag{5.34}$$

and

$$f_{\epsilon,\delta,t,\infty} \le f \le f^{\epsilon,\delta,t,\infty}\,. \tag{5.35}$$

Relation (5.31) implies that the function, $f_{\epsilon,\delta,t,\infty}$, is an increasing function of ϵ, δ, t^{-1}, and that the function, $f^{\epsilon,\delta,t,\infty}$, is a decreasing function of ϵ, δ, t^{-1}. Thus, the nonnegative function, $(f^{\epsilon,\delta,t,\infty} - f_{\epsilon,\delta,t,\infty})$, is a decreasing function of ϵ, δ, t^{-1}.

Since the function, $u_j+\delta$, is a continuous upper gradient, for $f_{\epsilon,\delta,t,j}$, it follows that $f_{\epsilon,\delta,t,j}$ is Lipschitz and $\mathrm{Lip}\, f_{\epsilon,\delta,t,j} \le u_j+\delta \le g+\delta$.

Claim: If for fixed δ, t, we successively let $j\to\infty$, $\epsilon\to 0$, then $f_{\epsilon,\delta,t,j}\,|\,C_t$ converges in L_p to $f\,|\,C_t$.

If we temporarily grant the above claim, then the proof can be completed as follows. Let successively, $j\to\infty$, $\epsilon\to 0$, $t\to\infty$, and finally, $\delta\to 0$. Take a suitable diagonal subsequence from the functions, $f_{\epsilon,\delta,t,j}$, and the corresponding subsequence from $\{u_j+\delta\}$. Then by Lebesgue's dominated convergence theorem, we obtain a sequence of Lipschitz functions, f_i, with continuous upper gradients, v_i, such that $f_i \xrightarrow{L_p} f$, $\limsup_{i\to\infty} \mathrm{Lip}\, v_i \le g$. This suffices to complete the proof.

Proof of claim. Since $C_t \subset \cup_k B_{2\epsilon}(z_{\epsilon,t,k})$, from (5.34), (5.35) and the definition of $f_{\epsilon,\delta,t,\infty}, f^{\epsilon,\delta,t,\infty}$, we get

$$\int_{C_t} |f - f^{\epsilon,\delta,t,j}|^p\, d\mu \le 2\sum_k \int_{B_{2\epsilon}(z_{\epsilon,t,k})} \big(F_{g+\delta}(z_{\epsilon,t,k}, z)\big)^p d\mu\,. \tag{5.36}$$

By a standard consequence of (0.1), the multiplicity of the covering, $\{B_{2\epsilon}(z_{\epsilon,t,k})\}$, is at most $2^{4\kappa}$. Thus, from (5.36) and (5.28) (with $r=2\epsilon$) we get

$$\sum_k \int_{B_{2\epsilon}(z_{\epsilon,t,k})} \big(F_{g+\delta}(z_{\epsilon,t,k}, z)\big)^p d\mu \le 2^{4\kappa+1}\big((2^{4\kappa}+1)\tau 2\epsilon(\delta+t)\big)^p \mu(B_{2d}(\overline{z}))\,. \tag{5.37}$$

The inequalities, (5.36) and (5.37), imply our claim. □

REMARK 5.38. Note that it follows from (5.28), that for all $g, \underline{z}$ such that $M_p g(\underline{z}) < \infty$, the point, $\underline{z}$, is a Lebesgue point of the function $F_g(\underline{z}, z)$.

6 Doubling and Poincaré Implies $g_f = \mathrm{Lip}\, f$

In this section, we remove from Theorem 5.1, the hypothesis that Z is a length space.

Theorem 6.1. *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Then* $g_f(\underline{z}) = \mathrm{Lip}\, f(\underline{z})$, *for* μ*-a.e.* $\underline{z}$.

Let ρ denote the underlying metric on Z and let ρ_0 denote the canonically associated length space metric as in section 17. Denote by $\mathrm{Lip}_{\rho} f$, g_f^{ρ}, respectively $\mathrm{Lip}_{\rho_0} f$, $g_f^{\rho_0}(\underline{z})$ the pointwise Lipschitz constant and minimal upper gradient for the metrics, ρ, ρ_0.

Theorem 6.1 is a direct consequence of the following proposition.

Proposition 6.2. . *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Then* $\mathrm{Lip}_{\rho} f(\underline{z}) = \mathrm{Lip}_{\rho_0} f(\underline{z})$, *for* μ*-a.e.* $\underline{z}$.

Proof of Theorem 6.1. We have $g_f^{\rho}(\underline{z}) \leq \mathrm{Lip}_{\rho} f(\underline{z}) = \mathrm{Lip}_{\rho_0} f(\underline{z}) = g_f^{\rho_0}(\underline{z})$, for μ-a.e. $\underline{z}$, where the last equality is the content of Theorem 5.1. However, since $\rho \leq \rho_0$, it follows that $g_f^{\rho_0}(\underline{z}) \leq g_f^{\rho}(\underline{z})$, for μ-a.e. $\underline{z}$; compare Remark 2.14. This suffices to complete the proof. □

The strategy for proving Proposition 6.2 is analogous to that used in the proof of Theorem 5.1. We first prove Proposition 6.2 for the case in which f is a distance function, $\rho_0(\underline{z}, z)$. Then we use an approximation theorem (Theorem 6.5) together with Theorem 4.53.

Lemma 6.3. *Let* (Z,μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Then* $\mathrm{Lip}_{\rho}\, \rho_0(\underline{z}) = 1 = \mathrm{Lip}_{\rho_0}\, \rho_0(\underline{z})$, *for* μ*-a.e.* $\underline{z}$.

Proof. We have $\rho_0(\underline{z}, z) = \lim_{\epsilon\to 0} \rho_{\underline{z},\epsilon}(z)$, where $\rho_{\underline{z},\epsilon}$ is the function defined in Theorem 17.1. As noted there, $\mathrm{Lip}_{\rho}\, \rho_{\underline{z},\epsilon}(z) \leq 1$, for all $\underline{z}, z, \epsilon$. Since $\mathrm{Lip}_{\rho_0}\, \rho_0(z) \equiv 1$, our assertion follows from Theorem 4.53. □

Remark 6.4. In the application to the proof of Proposition 6.2, we will take the underlying metric in the following Theorem 6.5 to be the length space metric ρ_0. In section 10, we will need a more general formulation which allows for λ-quasi-convex metrics. Although this extension is essentially trivial, we just write the proof in the quasi-convex case in order to simplify the exposition; compare Remark 6.23.

As usual, let $\mathbf{Lip}\, f$ denote the Lipschitz constant of f.

Theorem 6.5. *Let* (Z,μ) *satisfy* (0.1) *with* Z *quasi-convex. Let* $f : B_R(\overline{z}) \to \mathbf{R}$ *be Lipschitz, where* $0 < R < \infty$. *Then there exists a sequence of Lipschitz functions,* $f_i : B_R(\overline{z}) \to \mathbf{R}$, *and for each* i, *a finite collection of pointed closed sets,* $z_{i,\ell} \in C_{i,\ell} \subset B_R(\overline{z})$, *and constants,* $0 \leq c_{i,\ell}$, *such that*

$$f_i \xrightarrow{\text{unif.}} f\,, \tag{6.6}$$

$$\mathbf{Lip}\, f_i \to \mathrm{Lip}\ f\,, \tag{6.7}$$

$$\operatorname{Lip} f_i \xrightarrow{L_p} \operatorname{Lip} f \qquad (1 < p < \infty)\,. \tag{6.8}$$

$$f_i(z)\,|\,C_{i,\ell} = c_{i,\ell}\overline{z_i, z}\,, \tag{6.9}$$

$$\lim_{i\to 0} \mu\big(B_R(\overline{z}) \setminus \cup_\ell C_{i,\ell}\big) = 0\,. \tag{6.10}$$

Proof of Proposition 6.2. We apply Theorem 6.5, to Z equipped with the metric ρ_0. Define the function h_i by

$$h_i(z) = \begin{cases} \operatorname{Lip}_{\rho_0} f_i(z) & z \in \cup_\ell C_{i,\ell}, \\ \operatorname{Lip}_{\rho} f_i(z) & z \in B_d(\overline{z}) \setminus \cup_\ell C_{i,\ell}\,. \end{cases} \tag{6.11}$$

By (6.9), (6.11) together with Lemma 6.3, we have $\operatorname{Lip}_\rho f_i(\underline{z}) \le h_i(\underline{z})$, for μ-a.e. $\underline{z}$. From (6.7), (6.10), (6.11), we get $\limsup_i h_i(\underline{z}) \le \operatorname{Lip}_{\rho_0} f(\underline{z})$, for μ-a.e. $\underline{z}$. By Theorem 4.53, which we apply with $||df(z)|| = \operatorname{Lip}_\rho f(z)$, this suffices to complete the proof. □

The proof of Theorem 6.5 will require a number of lemmas. We begin by recalling a property of doubling measures on length spaces. The following assertion is in the spirit of relative volume comparison for riemannian manifolds with a definite lower Ricci curvature bound, although weaker.

Proposition 6.12 ([CoMi2]). *Let (Z,μ) satisfy (0.1) and assume that Z is a length space. Then for all z, s and $0 < \delta < 1$,*

$$\mu\big(B_s(z) \setminus B_{(1-\delta)s}(z)\big) \le (2\delta)^\beta \mu(B_s(z))\,, \tag{6.13}$$

where

$$\beta = \log_2(1 + 2^{-5\kappa})\,. \tag{6.14}$$

In connection with Lemma 6.15 below, note that the right-hand side of (6.18) can be made arbitrarily small by taking δ sufficiently small and N sufficiently large.

Lemma 6.15. *Let (Z,μ) satisfy (0.1), with Z a length space. Let $A \subset B_{R+r}(\bar{z})$ be a measurable subset. Then for all $0 < \delta < 1$, $1 \le N < \infty$ and β as in (6.14), there exists a finite collection of balls, $\mathcal{B} = \{B_{r_j}(z_j)\}$, such that*

$$\tfrac{1}{2}\delta^{-N} r \le r_j \le r\,, \tag{6.16}$$

$$\overline{B_{r_{j_1}}(z_{j_1}), B_{r_{j_2}}(z_{j_2})} \ge \tfrac{1}{2}\min(r_{j_1}, r_{j_2}) \qquad (j_1 \ne j_2)\,, \tag{6.17}$$

$$\mu\big(A \setminus \cup_j B_{r_j}(z_i)\big) \le \left((2\delta)^\beta + \frac{1}{2^{-2\kappa}(1-(2\delta)^\beta)N}\right) \mu\big(B_{R+r}(\bar{z})\big)\,. \tag{6.18}$$

Proof. Choose a maximal set of points, $z_{j,0} \in A$, of cardinality, $n_0 < \infty$, such that distinct points lie at distance at least r. Put $F_0 = A$, $E_0 = \cup_j B_{\frac{1}{4}r}(z_{j,0})$ and $F_1 = A \setminus E_0$.

Next, choose a maximal set of points, $z_{j,1} \in F_1$, of cardinality, $n_1 < \infty$, such that distinct points lie at distance at least δr. Put $E_1 = \cup_j B_{\frac{1}{4}\delta r}(z_{j,1})$ and $F_2 = A \setminus (E_0 \cup E_1)$. Proceeding by induction, we define sets, E_k, F_{k+1}, for $0 \le k \le N-1$, with $E_{k-1} = \cup_j B_{\frac{1}{4}\delta^{k-1}r}(z_{j,k-1})$ and $F_k = A \setminus (E_0 \cup \cdots \cup E_{k-1})$.

By (0.1), we have

$$\mu(F_N) \le \mu(F_k) \le 2^{-2\kappa}\mu(E_k)\,. \tag{6.19}$$

The collection of balls, $\mathcal{B} = \{B_{\frac{1}{4}\delta^k r}(z_{j,k-1})\}$, where $1 \le k \le N$, $1 \le j \le n_{k-1}$, satisfies (6.16), (6.17). Since these balls are mutually disjoint, we get

$$\mu\big(\cup_{j,k} B_{\frac{1}{4}\delta^k r}(z_{j,k-1})\big) \le \mu\big(B_{R+r}(\bar z)\big)\,. \tag{6.20}$$

Thus, from (6.13) and the definition of F_N, we obtain

$$\mu\big(F_N \setminus \cup_{j,k} B_{\frac{1}{4}\delta^k r}(z_{j,k-1})\big) \le (2\delta)^\beta \mu\big(B_{R+r}(\bar z)\big)\,. \tag{6.21}$$

In (6.19), we put $k = 0, 1, \dots, N-1$ and add the resulting inequalities. Then by (6.17), (6.20), we get

$$\big(1 - (2\delta)^\beta\big) N\mu(F_N) \le 2^{-2\kappa}\mu\big(B_{R+r}(\bar z)\big)\,. \tag{6.22}$$

Relations (6.21), (6.22) imply (6.18). □

Remark 6.23. Note that given a λ-quasi-convex metric, Lemma 6.15 can be used to provide coverings by balls which are defined with respect to the canonically associated length space metric. With this remark, it is easy to modify the statements and proofs of Lemmas 6.24, 6.30 below (and hence, the proof of Theorem 6.5) so that they are valid in the quasi-convex case.

We now return to the context of Theorem 6.5.

Lemma 6.24. *Let (Z, μ) satisfy (0.1). If $f : B_R(\bar z) \to \mathbf{R}$ is Lipschitz, then for all $1 > \eta, \psi, \xi, \chi > 0$, there exists $0 < r_f = r_f(\eta, \psi, \xi, \chi, R, \bar z)$, and $Z_f = Z_f(\eta, \psi, \xi, \chi, R, \bar z) \subset B_R(\bar z)$, such that*

$$\mu(Z_f) \ge (1-\eta)\mu(B_R(\bar z))\,, \tag{6.25}$$

and for all $\underline{z} \in Z_f$,

$$\fint_{B_{3r}(\underline{z})} \big|\mathrm{Lip}\, f(z) - \mathrm{Lip}\, f(\underline{z})\big|^p d\mu < \psi \qquad (0 < r \le r_f)\,. \tag{6.26}$$

Moreover, for all $z_1, z_2 \in B_r(\underline{z})$, such that $\overline{z_1, z_2} > \xi r$,

$$\frac{|f(z_1) - f(z_2)|}{\overline{z_1, z_2}} < \mathrm{Lip}\ f(\underline{z}) + \chi \qquad (0 < r \le r_f)\,. \tag{6.27}$$

Proof. It follows from Lusin's theorem that there exists $r_1 > 0$, and $Z_1 \subset B_R(\bar z)$, with $\mu(Z_1) \ge \big(1 - \frac{1}{4}\eta\big)\mu(B_R(\bar z))$, such that for all $z_1, z_2 \in Z_1$, with $\overline{z_1, z_2} \le r_1$, we have $|\mathrm{Lip}\, f(z_1) - \mathrm{Lip}\, f(z_2)| < \frac{1}{4}\chi$. Moreover, it follows from the definition of $\mathrm{Lip}\, f$, that there exist $0 < r_2 \leq \frac{1}{2}r_1$ and $Z_2 \subset Z_1$, with $\mu(Z_2) \geq \big(1 - \frac{1}{2}\eta\big)\mu(B_R(\bar{z}))$, such that for all $\underline{z} \in Z_2$ and $\tilde{z}$, with $\overline{\underline{z}, \tilde{z}} \leq r_2$, we have

$$\frac{|f(\tilde{z}) - f(\underline{z})|}{\overline{\tilde{z}, \underline{z}}} < \mathrm{Lip}\, f(\underline{z}) + \frac{1}{4}\chi\,. \tag{6.28}$$

Thus, if $z_1, z_2 \in Z_2 \cap B_{r_2}(\underline{z})$, with $\underline{z} \in Z_2$, then

$$\frac{|f(z_1) - f(z_2)|}{\overline{z_1, z_2}} < \mathrm{Lip}\, f(\underline{z}) + \frac{1}{2}\chi\,. \tag{6.29}$$

Let L denote the Lipschitz constant of f. We can assume $\frac{1}{4}\chi < L$. By (0.1) and the Lebesgue differentiation theorem, there exists $0 < r_3 \leq r_2$ and $Z_3 \subset Z_2$, with $\mu(Z_3) \geq \big(1 - \frac{3}{4}\eta\big)\mu(B_R(\bar{z}))$, such that for all $\underline{z} \in Z_3$ and $r \leq r_3$, the set, $Z_3 \cap B_{r_3}(\underline{z})$, is $\frac{\xi\chi}{8L}r$-dense in $B_r(\underline{z})$. It follows easily that (6.27) holds, for all $z_1, z_2 \in B_r(\underline{z})$.

Finally, by the Lebesgue differentiation theorem, there exists $0 < r_f \leq r_3$ and $Z_f \subset Z_3$, with $\mu(Z_f) \geq (1-\eta)\mu(B_R(\bar{z}))$, such that for all $\underline{z} \in Z_3$, (6.26) holds. This suffices to complete the proof. □

Let $r_f = r_f(\eta, \psi, \xi, \chi, R, \bar{z})$ be as in Lemma 6.24. From Lemma 6.24 together with Lemma 6.15, we obtain:

Lemma 6.30. *Let (Z, μ) satisfy (0.1), with Z a length space. Let $f : B_R(\bar{z}) \to \mathbf{R}$ be a Lipschitz. Given $\eta, \psi, \xi, \chi > 0$, there exists a finite collection of mutually disjoint balls, $\{B_{r_j}(z_j)\}$, with $z_j \in B_R(\bar{z})$, $r_j \leq 1$, for all j, such that*

$$\overline{B_{r_i}(z_j), B_{r_\ell}(z_\ell)} \geq \tfrac{1}{2}(r_j + r_\ell) \qquad (j \neq \ell)\,, \tag{6.31}$$

$$\mu\big(B_R(\overline{z}) \setminus \cup_j B_{r_j}(z_j)\big) < C(\kappa)\,\eta\,\mu\big(B_R(\bar{z})\big)\,, \tag{6.32}$$

$$C(\kappa, \eta) r_f \leq r_j \leq r_f\,, \tag{6.33}$$

$$\bigg(⨍_{B_{3r_j}(z_j)} |\mathrm{Lip}\, f(z) - \mathrm{Lip}\, f(z_j)|^p \, d\mu\bigg)^{1/p} < \psi\,. \tag{6.34}$$

Moreover, for all z_j and $\underline{z}_{1,j}, \underline{z}_{2,j} \in B_{r_j}(z_j)$, such that $\overline{\underline{z}_{1,j}, \underline{z}_{2,j}} > \xi r_j$, we have

$$\frac{|f(\underline{z}_{1,j}) - f(\underline{z}_{2,j})|}{\overline{\underline{z}_{1,j}, \underline{z}_{2,j}}} < \mathrm{Lip}\ f(z_j) + \chi\,. \tag{6.35}$$

In addition to the results established so far, the following proof of Proposition 6.2 depends on MacShane's lemma; see (8.3).

Proof of Proposition 6.2. For each fixed $\eta, \psi, \xi > 0$ as in Lemma 6.30, we will construct a function, $f_\# : B_R(\bar{z}) \to \mathbf{R}$. For $i = 1, 2, \ldots$, we will take $\eta = \psi = \chi = i^{-1}$, choose $\xi = \xi(\eta)$ sufficiently small and set $f_i = f_\#$.

It will be clear that the resulting sequence of functions has the required properties.

Fix $\eta, \psi, \xi > 0$. Let $\{B_{r_j}(z_j)\}$ be the covering provided by Lemma 6.30. For each ball, $B_{r_j}(z_j)$, choose a maximal set of points, $\{w_{j,k}\}$, such that if $k \neq k'$, then $w_{j,k}$ and $w_{j,k'}$ are at mutual distance at least $2\xi r_j$.

For all j, k, define the function, $f_\#$, on the set, $\cup_{j,k} w_{j,k}$, by setting $f_\#(w_{j,k}) = f(w_{j,k})$.

Next, for each j, on the ball, $B_{r_j}(z_j)$, define $f_\#$ to be the canonical extension given by (8.3), of the function, $f_\#$, on $\cup_k w_{j,k} \subset B_{r_j}(z_j)$.

Finally, define $f_\#$ on the whole of Z to be the canonical extension given by (8.3) (MacShane's lemma) of the function, $f_\#$, on the set, $\cup B_{r_j}(z_j) \subset Z$.

If we define the functions, f_i, as above, then it is completely straightforward to check that these functions have the required properties. □

From Theorem 6.1, we can deduce the following corollary, which plays a significant role in the proof of infinitesimal generalized linearity for our generalization of Rademacher's theorem; see Theorem 10.2.

COROLLARY 6.36. *Let (Z, μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. If f is Lipschitz, then for μ-a.e. $\underline{z} \in Z$, we have existence of the limit,*

$$\begin{aligned} Lip\, f(\underline{z}) &= \operatorname{Lip} f(\underline{z}) \\ &= \lim_{r \to 0} \sup_{z \in \partial B_r(\underline{z})} \frac{|f(z) - f(\underline{z})|}{\overline{z, \underline{z}}} \,. \end{aligned} \tag{6.37}$$

Proof. As noted in Proposition 1.11, the function, *Lip* f, defined in (1.10), is an upper gradient for f. Thus, the corollary follows from Theorem 6.1. □

As in Remark 2.29, in our next application of Theorem 6.1, we write $g_{f,p}$ to indicate the possible dependence of a minimal upper gradient on the choice of p.

COROLLARY 6.38. *Let (Z, μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. If f is locally in $H_{1,p'}$ for some $p' \geq p$, then $g_{f,p'} = \operatorname{Lip} f$. In particular, the sets, $V_\alpha(f)$, of Theorem 4.38, can be chosen such that the conclusions of that theorem hold, no matter which value, $p' \in [p, \infty)$, is used in defining the notion of asymptotic generalized linearity.*

Proof. To see the second assertion, choose a countable dense subset $\{p'_i\} \subset [p, \infty)$. It is clear that the sets, $U_\alpha, V_\alpha(f)$, can be chosen such that the assertions of Theorem 4.24 hold when the notion of asymptotic generalized linearity is defined relative to any of values, p'_i, lying in this subset. Given that, $g_{f,p'} = g_{f,p}$, for all $p' \geq p$, our second assertion follows by a straightforward continuity argument. □

7 p-Harmonic Functions and the Dirichlet Problem

In this section, we define p-harmonic functions and prove some of their basic properties.

In discussing existence of solutions to the Dirichlet problem, we first consider the case in which U is bounded and $f \in H_{1,p}(U)$ is essentially bounded. Assuming neither (0.1), nor (4.3), we show that for $1 < p < \infty$, there is a bounded solution to the Dirichlet problem in the sense of the relaxed topology; see Definition 7.3.

Given (0.1), (4.3), we show that a solution exists in the weak sense of traces for arbitrary, U, $f \in H_{1,p}(U)$ and $1 < p < \infty$. Here, we appeal to Theorem 4.48.

We show in general that if $\mathbf{b}_1, \mathbf{b}_2$ are two solutions to the same relaxed Dirchlet problem, then $g_{\mathbf{b}_2 - \mathbf{b}_1} \equiv 0$. Thus, if the Dirichlet Poincaré inequality holds, the solution is unique; compare (7.15). Finally, we prove a version of the maximum principle.

Definition 7.1. A function, $\mathbf{b} : U \to \mathbf{R}$ is *p-harmonic* if $\mathbf{b}\,|\,V$ is in $H_{1,p}(V)$, for every bounded open set, $V \subset U$, and for every $k \in \mathcal{K}(V)$, the function, $\mathbf{b}\,|\,V$, satisfies

$$|g_{\mathbf{b}+k}|_{L_p} \geq |g_{\mathbf{b}}|_{L_p} . \tag{7.2}$$

Note that if $\mathbf{b}$ is p-harmonic, it follows that (7.2) actually holds for all $\bar{k} \in \overset{\circ}{H}_{1,p}(V)$. Moreover, if (7.2) holds for some V, then it holds for all $V' \subset V$. For $A \subset H_{1,p}(U)$, denote by $\tilde{A}$, the closure of A in the relaxed topology. Clearly, $\tilde{A}$ is closed in $H_{1,p}(U)$, for any A. Moreover, we have $\bar{A} \subset \tilde{A} = \bar{\tilde{A}}$, where $\bar{A}$ denotes the closure of A in the usual (norm) topology of $H_{1,p}$.

Definition 7.3. Let $f \in H_{1,p}(U)$. A function, $\mathbf{b} : U \to \mathbf{R}$ is a *relaxed* solution to the Dirichlet problem for f, if $\mathbf{b}$ is p-harmonic and $\mathbf{b} \in f + \tilde{\mathcal{K}}$.

In order to be certain that Definition 7.3 to be a reasonable one, we must show that $\widetilde{f + \mathcal{K}} = f + \widetilde{\mathcal{K}}$, for all $f \in H_{1,p}(U)$. In this case, it follows directly, that $\inf_{k \in \mathcal{K}} |g_{f+k}|_{L_p} = |g_{\mathbf{b}}|_{L_p}$ and $|g_{\mathbf{b}}|_{L_p} \leq |g_{\mathbf{b}+\tilde{k}}|_{L_p}$, for all $\tilde{k} \in \tilde{\mathcal{K}}$. To this end (and as an aid to proving the existence of relaxed solutions to the Dirichlet problem by a direct method) we will introduce an additional notion.

Definition 7.4. Let $f_i \in H_{1,p}(U)$, for all i. The sequence, $\{f_i\}$, converges to f in the *very relaxed topology* of $H_{1,p}$, if $f_i \xrightarrow{L_p} f$ and there exists a sequence of generalized upper gradients, g_i for f_i, such that $g_i \xrightarrow{L_p} g$, for some $g \in L_p$.

For $A \subset H_{1,p}(U)$, denote by $\hat{A}$, the closure of A in the very relaxed topology. Clearly, $\hat{\mathcal{V}}$ is a subspace, for any subspace, $\mathcal{V} \subset H_{1,p}(U)$.

LEMMA 7.5. *For all A, we have $\tilde{A} \subset \hat{A}$.*

Proof. This follows immediately from Proposition 2.13. □

LEMMA 7.6. *If $f \in H_{1,p}(U)$, then $\widehat{f + \mathcal{V}} = f + \hat{\mathcal{V}}$.*

Proof. Let $u \in \widehat{f + \mathcal{V}}$. Let $f + f_i \xrightarrow{L_p} u$, $g_i \xrightarrow{L_p} g$, where g_i is a generalized upper gradient for $f + f_i$. Since, $f_i = f + f_i - f$, it follows that $g_i + g_f$ is a generalized upper gradient for f_i. From this, we get $\widehat{f + \mathcal{V}} \subset f + \hat{\mathcal{V}}$ and the opposite inclusion follows similarly. □

PROPOSITION 7.7. *The equality, $\widehat{f + \mathcal{K}} = \widetilde{f + \mathcal{K}}$ holds. In particular, $\widetilde{f + \mathcal{K}} = f + \tilde{\mathcal{K}}$ is a closed affine subspace of $H_{1,p}(U)$.*

Proof. By Lemmas 7.5, 7.6, it suffices to show that $\widehat{f + \mathcal{K}} \subset \widetilde{f + \mathcal{K}}$. The proof is similar to that of Proposition 2.17. If $u \in \widehat{f + \mathcal{K}}$, there exists $k_i \in \mathcal{K}$, such that $f + k_i \xrightarrow{L_p} u$, and generalized upper gradients, g_i, for $f + k_i$, such that $g_i \xrightarrow{L_p} g$, for some $g \in L_p$.

Let $\eta > 0$ and and let $\phi : U \to [0,1]$ be a Lipschitz function with $\operatorname{supp}\phi \subset U$, such that $\phi \,|\, U_\eta \equiv 1$. The function, $u_{i,\eta} = \phi u + (1-\phi)(f + k_i)$, lies in $f + \mathcal{K}$ and $u_{i,\eta} \xrightarrow{L_p} u$, as $\eta \to 0$. Also, as follows easily from Lemma 1.7, for all $\epsilon > 0$, the function, $g_{i,\epsilon,\eta} = \operatorname{Lip}(1-\phi)(|u - (f + k_i)| + \epsilon)$ $+(1-\phi+\epsilon)(g_u + g_i) + g_u$ is a generalized upper gradient for $u_{i,\eta}$. By choosing suitable diagonal sequences, $u_{i(\eta),\eta}$, $g_{i(\eta),\epsilon(i(\eta),\eta),\eta}$, such that $u_{i(\eta),\eta} \xrightarrow{L_p} u$, $g_{i(\eta),\epsilon(i(\eta),\eta),\eta} \xrightarrow{L_p} g_u$, as $\eta \to 0$, the proof is completed. □

Theorem 7.8 (Existence of relaxed solutions)**.** *For all $1 < p < \infty$ and bounded open sets, U and essentially bounded $f \in H_{1,p}(U)$, the Dirichlet problem for f has a relaxed solution, $f + \tilde{k}$, such that* $\operatorname{ess\,sup} f + \tilde{k} \leq \operatorname{ess\,sup} f$ *and* $\operatorname{ess\,inf} f + \tilde{k} \geq \operatorname{ess\,inf} f$.

Proof. Let $\{k_i\}$ be a sequence in $\mathcal{K}$ such that $|g_{f+k_i}|_{L_p} \to \liminf_{k \in \mathcal{K}} |g_{f+k}|_{L_p}$. Since the set, $f + \mathcal{K}$, is convex, it follows from the uniform convexity of L_p, for $1 < p < \infty$ that $g_{f+k_i} \xrightarrow{L_p} g$, for some $g \in L_p$. Without loss of generality, we can assume

$$|g_{f+k_i} - g|_{L_p} \leq 2^{-i} \,. \tag{7.9}$$

Since truncation does not increase the size of the generalized upper gradient, without loss of generality, we can assume that the functions, $f + k_i$ are uniformly bounded in absolute value. Since U is bounded, it follows from the monotone convergence theorem that for fixed i, the sequence, $f_{i,j} = \{\min(f + k_i, \ldots, f + k_j)\}$, converges in L_p to a function, $f_{i,\infty}$. Similarly, the increasing sequence, $\{f_{i,\infty}\}$ converges in L_p to a function, $\mathbf{b}$.

It follows from (7.9) and Corollary 2.26, that a for a suitably chosen diagonal sequence, $f_{i,j(i)}$, we have, $f_{i,j(i)} \xrightarrow{L_p} \mathbf{b}\,, g_{f_{i,j(i)}} \xrightarrow{L_p} g$. Thus, by Proposition 7.7, together with the discussion following Definition 7.3, we conclude that $\mathbf{b}$ is a relaxed solution to the Dirichlet problem for f. □

Definition 7.10. Given $f \in H_{1,p}(U)$, the function, $\mathbf{b} \in H_{1,p}(U)$, is a solution to the Dirichlet problem for f in the *weak sense of traces*, if $\mathbf{b} = f + \overline{k}$ is p-harmonic and $\overline{k} \in \overset{\circ}{H}_{1,p}(U)$.

Remark 7.11. Set $\partial U = \overline{U} \setminus U$. If U is bounded and $u : \partial U \to \mathbf{R}$ is Lipschitz, then there exists a Lipschitz extension of f to $\overline{U}$; see (8.2), (8.3). Since μ is finite on bounded sets, it follows that such an extension is in $H_{1,p}$. Since μ is Borel regular, we can use an exhaustion of U by closed bounded subsets, together with (8.2) or (8.3), to show that any two such extensions differ by an element of $\overset{\circ}{H}_{1,p}$. Thus, there is a well defined notion of a solution to the Dirichlet problem with boundary values u (in the weak sense of traces.)

Theorem 7.12 (Existence of solutions in the weak sense of traces)**.** *Let (Z,μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Then for $f \in H_{1,p}(U)$, the Dirichlet problem for f has a solution in the weak sense of traces.*

Proof. Let $f+k_i$ be as the proof of Theorem 7.8. By applying Theorem 4.50, we can assume that this sequence converges in $H_{1,p}(U)$. This suffices to complete the proof. □

Remark 7.13. The previous argument should be compared to that of [HeKM].

Theorem 7.14 (Uniqueness)**.** *Let U be bounded and let $f \in H_{1,p}(U)$, where $1 < p < \infty$.*

i) *If $\mathbf{b}_1, \mathbf{b}_2$, are relaxed solutions to the Dirichlet problem for f, then $g_{\mathbf{b}_1-\mathbf{b}_2}(z) = 0$, for μ-a.e. z.*

ii) *Given in addition, the Dirichlet Poincaré inequality,*

$$\int_U |k|^p \, d\mu \le \tau^p \int_U (g_k)^p \, d\mu \qquad (k \in \mathcal{K})\,, \tag{7.15}$$

then solutions to the Dirichlet problem for f are unique.

Proof. Assume that $\mathbf{b}_1, \mathbf{b}_2$ are two relaxed solutions corresponding to the same $f \in H_{1,p}(U)$. Since, U is bounded, it follows from the strict convexity of L_p, that $g_{\mathbf{b}_1} = g_{\mathbf{b}_2}$, μ-a.e.

The function, $\mathbf{b} = \max(\mathbf{b}_1, \mathbf{b}_2)$, satisfies $\mathbf{b} - f \in \tilde{\mathcal{K}}$. Thus, by Corollary 2.26, $\mathbf{b}$ is a relaxed solution to the Dirichlet problem for f as well.

Let $V \subset U$, denote the subset of points, z, such that $\mathbf{b}_2(z) < \mathbf{b}(z)$, and $g_{\mathbf{b}}(z) > 0$.

We claim that $\mu(V) = 0$. If not, then for some $c \in (-\infty, \infty)$, the subset, $W_c \subset V$, on which $\mathbf{b}_2 < c < \mathbf{b}$ has positive measure. Let $\hat{\mathbf{b}} : U \to \mathbf{R}$ be defined by

$$\hat{\mathbf{b}}(z) = \begin{cases} \mathbf{b}(z) & \text{if } \mathbf{b} \le c\,, \\ c & \text{if } \mathbf{b}_2 < c < \mathbf{b}\,, \\ \mathbf{b}_2(z) & \text{if } c \le \mathbf{b}_2\,. \end{cases} \tag{7.16}$$

It is easy to see that $\hat{\mathbf{b}} - f \in \tilde{\mathcal{K}}$. Since $g_{\mathbf{b}} > 0$ on W_c and $\mu(W_c) > 0$, it follows from Corollaries 2.25, 2.26, that $|g_{\hat{\mathbf{b}}}|_{L_p} < |g_{\mathbf{b}}| = |g_{\mathbf{b}_2}| = 0$. This contradicts the assumption that $\mathbf{b}$ is a relaxed solution to the Dirichlet problem for f.

Thus, for μ-a.e. $\underline{z} \in U$, we have either $\mathbf{b}_1(\underline{z}) = \mathbf{b}_2(\underline{z})$, or $g_{\mathbf{b}_1}(\underline{z}) = g_{\mathbf{b}_2}(\underline{z}) = 0$. By Proposition 2.22, we get $g_{\mathbf{b}_1 - \mathbf{b}_2}(\underline{z}) = 0$, for μ-a.e. $\underline{z}$ such that $\mathbf{b}_1(\underline{z}) = \mathbf{b}_2(\underline{z})$. Since, $g_{\mathbf{b}_1 - \mathbf{b}_2}(\underline{z}) \le g_{\mathbf{b}_1} + g_{\mathbf{b}_2}(\underline{z})$, we also get $g_{\mathbf{b}_1 - \mathbf{b}_2}(\underline{z}) = 0$, for μ-a.e. $\underline{z}$ such that $g_{\mathbf{b}_1}(\underline{z}) = g_{\mathbf{b}_2}(\underline{z}) = 0$. This suffices to complete the proof. □

Theorem 7.17 (Maximum principle)**.** *Let U be a bounded open set and let $1 < p < \infty$. Let $u_1, u_2 \in H_{1,p}(U)$ satisfy $u_1 \le u_2$, and let $\mathbf{b}_1, \mathbf{b}_2$ be corresponding relaxed solutions to the Dirichlet problems. Then either* $\operatorname{ess\,sup} \mathbf{b}_2 - \mathbf{b}_1 \ge 0$ *or the solution to at least one of these Dirichlet problems is not unique and $g_{\mathbf{b}_1}(z) = g_{\mathbf{b}_2}(z) = 0$, for μ-a.e. z such that $\mathbf{b}_1(z) > \mathbf{b}_2(z)$. In particular, if* (7.15) *holds, then* $\operatorname{ess\,sup} \mathbf{b}_2 - \mathbf{b}_1 \ge 0$.

Proof. Since the solutions for the Dirichlet problem with boundary values, $u_1 - c$, are precisely those of the form $\mathbf{b}_1 - c$, where $\mathbf{b}_1$ is a solution with boundary values, u_1, we can assume without loss of generality that $u_1 < u_2$. If the set of points, z, such that $\mathbf{b}_1(z) > \mathbf{b}_2(z)$ has positive measure, then at least one of the functions, $\min(\mathbf{b}_1, \mathbf{b}_2)$, $\max(\mathbf{b}_1, \mathbf{b}_2)$ provides a solution to at least one of our Dirichlet problems which differs on a set of positive measure from the corresponding solution, $\mathbf{b}_1$, respectively, $\mathbf{b}_2$. The remainder of our assertions follow directly from the arguments given in the proof of Theorem 7.14. □

REMARK 7.18. If in particular, the Poincaré inequality, (4.3), holds, then those conclusions of Theorems 7.14, 7.17 which obtain in case the Dirichlet Poincaré inequality holds, are valid for balls whose closures are properly contained in Z; see (4.5).

REMARK 7.19. If (0.1), (4.3), (7.15), hold, then the previous discussion can be adapted so as to apply to p-harmonic functions defined with respect to an L_∞ riemannian metric as in section 4, or more generally an L_∞ Finsler metric. By the latter, we mean that the unit sphere in the cotangent space is smooth and strictly convex. For such metrics, the discussion of [HeKM] shows that p-harmonic functions, $\mathbf{b}$, are α-Hölder continuous, for any $1 < p < \infty$ (where $\alpha = \alpha(\kappa, \tau, \lambda, p)$); compare Remark 4.60. In this connection we mention that the assumption that unit sphere is smooth and strictly convex enters in deriving Caccioppoli estimates on the function, $1/\mathbf{b}$ (whereas for the function $\mathbf{b}$ itself, it can be avoided).

8 Generalized Linear Functions

In this section, we show that given (0.1), (4.3), a generalized linear function, ℓ, has a canonical representation in terms of its boundary values; see Theorem 8.5. From this representation, it follows in particular, that any point lies on some geodesic line, γ, such that $\ell(\gamma(s)) = \operatorname{Lip} \ell \cdot s$, where s denotes arclength. Thus, if the gradient, $\nabla \ell$ were actually defined, then γ would be an integral curve of $\nabla \ell$.

Fix $1 < p < \infty$.

DEFINITION 8.1. A Lipschitz function, ℓ, is *generalized linear* if

i) Either $\ell \equiv 0$ or $\operatorname{range} \ell = (-\infty, \infty)$.

ii) ℓ is p-harmonic.

iii) $g_\ell \equiv c$, for some c.

The terminology notwithstanding, we can not assert in general that a linear combination of linear functions is again a linear function. None-theless, in section 11, spaces of generalized linear functions will arise naturally. If (0.1), (4.3) hold, then by Lemma 4.37, there is a definite bound on the dimension of any space of functions all of whose elements are generalized linear.

We will need some easily verified, though quite basic, general facts; compare Lemma 5.16.

Let $\mathbf{Lip}\, f$ denote the Lipschitz constant of f.

If Z is a metric space and $f : A \to \mathbf{R}$, is Lipschitz, for some $A \subset Z$, then the function, f^*, defined by

$$f^*(z) = \inf_{z' \in A} \left(f(z') + \mathbf{Lip}\, f \cdot \overline{z, z'} \right), \tag{8.2}$$

satisfies $f^* \,|\, A = f \,|\, A$ and $\mathbf{Lip}\, f^* = \mathbf{Lip}\, f$.

Similarly, the function, f_*, defined by

$$f_*(z) = \sup_{z' \in A} \left(f(z') - \mathbf{Lip}\, f \cdot \overline{z, z'} \right), \tag{8.3}$$

satisfies $f_* \,|\, A = f \,|\, A$ and $\mathbf{Lip}\, f_* = \mathbf{Lip}\, f$.

The above statements are sometimes referred to as 'MacShane's Lemma'.

If $\tilde f$ is any function satisfying, $\tilde f \,|\, A = f \,|\, A$ and $\mathbf{Lip}\, \tilde f = \mathbf{Lip}\, f$, then it is easy to verify that

$$f_* \le \tilde f \le f^* . \tag{8.4}$$

Theorem 8.5. *Let (Z, μ) satisfy* (0.1), *and* (4.3), *for some $1 < p < \infty$, with Z complete. Let $U \subset B_r(\underline{z}) \subsetneq Z$, and let $\ell : \overline{U} \to \mathbf{R}$ satisfy* ii), iii), *of Definition 8.1. Let ℓ^*, ℓ_* be as* (8.2), (8.3), *where $A = \partial U$. Then $\ell^* = \ell = \ell_*$.*

Proof. Since the functions, ℓ^*, ℓ_*, ℓ are all solutions to the Dirichlet problem the claim follows from Theorem 7.14 and Remark 7.18, together with Theorem 6.1 □

Theorem 8.6. *Let (Z, μ) satisfy* (0.1), *and* (4.3), *for some $1 < p < \infty$, with Z complete. Let $Z \setminus \overline{B_s(\underline{z})} \neq \emptyset$ and $\ell : B_s(\underline{z}) \to \mathbf{R}$ satisfy* ii), iii), *of Definition 8.1. Let ℓ^*, ℓ_* be as* (8.2), (8.3), *where $A = \partial B_s(\underline{z})$. Then $\ell^* = \ell = \ell_*$ and for all $z \in B_s(\underline{z})$, there exist $z^*, z_* \in \partial B_s(\underline{z})$ such that*

$$\ell(z^*) = \ell(z) - \mathbf{Lip}\, \ell \cdot \overline{z, z'} , \tag{8.7}$$

and

$$\ell(z_*) = \ell(z) + \mathbf{Lip}\, \ell \cdot \overline{z, z'} . \tag{8.8}$$

Also, there exist minimal geodesic segment, $\gamma : [\ell(z) - \ell(z^*),\, \ell(z_*) - \ell(z)] \to B_s(z)$, with $\gamma(\ell(z) - \ell(z^*)) = z^*$, $\gamma(0) = z$, $\gamma(\ell(z_*) - \ell(z)) = z_*$, such that

$$\ell(\gamma(s)) = \ell(z) + s . \tag{8.9}$$

If Z is a length space, then there exists γ as above, for any z^*, z_* satisfying (8.7), (8.8).

Proof. Relations (8.7), (8.8) follow immediately from using the relation $\ell^* = \ell_* = \ell$, together with (8.2), (8.3).

If z^*, z_* are points satisfying (8.7), (8.8) for which there exist minimal segments, γ^* from z^* to z, and γ_*, from z to z_*, then it is easy to check that (when suitably parameterized) $\gamma = \gamma^* \cup \gamma_*$ has the required properties. Thus, γ as above exists for any z^*, z_*, if Z is a length space.

It will suffice to show the existence of z_* for which there exists γ_* as above; the case of z^* is similar. Fix $1 < N < \infty$ and put $s_N = (\ell(z_*) - \ell(z))/N$. Let $z_{*,1,N} \in \partial B_{s_N}(z)$ satisfy (8.8). Proceeding by induction, we obtain points, $z_{*,i,N} \in \partial B_{s_N}(z_{*,i-1,N})$, for all $1 \leq i \leq N$. It is clear that $z_* = z_{*,N,N}$ satisfies (8.8). If we let $N \to \infty$, then by an obvious limiting argument, we obtain the required point, z_* and geodesic segment γ_*. □

Theorem 8.6 has the following direct implication. Let f be Lipschitz. We say a curve, $c : [a, b] \to Z$, is an *integral curve* for g_f, if $f'(c(t)) = g_f^2(c(t))$. Recall that a *line* is a geodesic, $\gamma : (-\infty, \infty) \to Z$, each finite segment of which is minimal.

Theorem 8.10. *Let* (Z, μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$, *with* Z *complete, noncompact. Let* $\ell : Z \to \mathbf{R}$ *satisfy ii), iii) of Definition 8.1. Then* ℓ *is generalized linear; equivalently,* $\ell \equiv 0$, *or range*$(f) = (-\infty, \infty)$. *Moreover, for all* $\underline{z} \in Z$, *there exists a line* γ, *parameterized proportional to arclength, with* $\gamma(0) = \underline{z}$, *which is an integral curve for* $g_\ell = \operatorname{Lip} \ell$.

Proof. Given $\underline{z} \in Z$, let $z^{*,s}, z_{*,s}$ denote points on $\partial B_s(\underline{z})$ such that (8.9) hold and let γ_s denote the corresponding minimal geodesic segment from $z_{*,s}, \gamma_{*,s}$. By a standard compactness argument, there exists a sequence, $s_i \to \infty$, such that (when suitably parameterized proportional to arclength) γ_{s_i} converges uniformly on compact subsets to some curve γ. It is easy to check that γ is the required integral curve. □

Recall $\gamma : [0, \infty) \to Z$, is called a *ray*, if each finite segment of γ is minimal. Put $b_{\gamma,s}(z) = \overline{z, \gamma(s)} - s$, where as usual, s denotes arclength. By a standard argument based on the triangle inequality, on any compact subset, the functions $b_{\gamma,s}$ are uniformly bounded below (independent of s). In addition, $b_{\gamma,s_2} \leq b_{\gamma,s_1}$, if $s_1 \leq s_2$. The *Busemann function* associated to γ is the function, $b_\gamma(z) = \lim_{s\to\infty} b_{\gamma,s}(z)$.

If $\gamma : (-\infty, \infty) \to Z$ is a line, let $\underline{\gamma} = \gamma \mid [0, \infty)$ and define $-\underline{\gamma} : [0, \infty) \to Z$, by $-\underline{\gamma}(s) = \gamma(-s)$. As in the proofs of Theorems 8.5, 8.6, 8.10, from MacShane's lemma, we immediately obtain:

Theorem 8.11. *Let the assumptions be as in Theorem 8.10. Let γ (whose existence is guaranteed by Theorem 8.10) be a line with $\gamma(0) = \underline{z}$, which is an integral curve for g_f. Then*

$$\ell(\underline{z}) - \mathrm{Lip}\,\ell \cdot b_{\underline{\gamma}} \le \ell \le \ell(\underline{z}) + \mathrm{Lip}\,\ell \cdot b_{-\underline{\gamma}}\,. \tag{8.12}$$

In particular, if $Z = \mathbf{R}^n$, then $\ell(\underline{z}) - \mathrm{Lip}\,\ell \cdot b_{\underline{\gamma}} = \ell = \ell(\underline{z}) + \mathrm{Lip}\,\ell \cdot b_{-\underline{\gamma}}$, and in this case, the generalized linear function, ℓ, is linear.

Remark 8.13. Theorem 8.11 should be compared to the splitting theorem in riemannian geometry; see [ChGr].

Remark 8.14. If one assumes that generalized linear functions on $\mathbf{R}^n$ are of class, C^2, then one can see that such functions are linear by applying Bochner's formula. However, the argument which proves Theorem 8.11, does not rely on any such a priori assumptions concerning the differentiability of generalized linear functions. Note that such an argument is required, if we are to legitimately assert that our results provide a self contained proof of the classical Rademacher theorem for Lipschitz functions on $\mathbf{R}^n$.

We close this section with some simple quantitative observations which hold in case Z is a length space (satisfying no additional conditions). These are used in section 16.

Let $A \subset Z$ and assume that closed bounded subsets of A are compact. Given $z \in Z$, let the supremum and infimum in (8.2), (8.3) be realized by $z' = z^*$, $z' = z_*$, respectively. Let γ_*, γ^* be minimal geodesic segments parameterized by arclength from z to z_* respectively, z to z^*. Then for $0 \le \mathbf{Lip}\, f \cdot s \le f_*(z) - f_*(z_*)$, respectively, $0 \le \mathbf{Lip}\, f \cdot s \le f^*(z^*) - f^*(z)$, we have

$$f^*(\gamma^*(s)) = f^*(z) + \mathbf{Lip}\, f \cdot s\,. \tag{8.15}$$

respectively,

$$f_*(\gamma_*(s)) = f_*(z) - \mathbf{Lip}\, f \cdot s\,, \tag{8.16}$$

In addition, for fixed z, the distance from z to the level surfaces, $(f_*)^{-1}(f_*(z) - \mathbf{Lip}\, f \cdot s), (f^*)^{-1}(f^*(z) + \mathbf{Lip}\, f \cdot s)$, is in each case, s. The points $\gamma_*(s), \gamma^*(s)$ are closest points on their level surfaces to the point, z. Moreover, if in (8.15), (8.16), these points are replaced by any other such closest points, then the relations in (8.15), (8.16) continue to hold.

Under the same assumptions, the next lemma is now any easy exercise; the proof will be omitted.

Lemma 8.17. *Let $f : Z \to \mathbf{R}$ satisfy*

$$f^* \le f_* + \mathbf{Lip}\, f \cdot \psi\,. \tag{8.18}$$

If $0 \le f(z) - t \le \mathbf{Lip}\, f(\overline{z, A} - \psi)$, *respectively* $0 \le t - f(z) \le \mathbf{Lip}\, f(\overline{z, A} - \psi)$, *then*

$$\big|f(z) - t - \mathbf{Lip}\, f \cdot \overline{z, f^{-1}(t)}\big| \le 2\mathbf{Lip}\, f \cdot \psi \,, \tag{8.19}$$

respectively,

$$\big|t - f(z) - \mathbf{Lip}\, f \cdot \overline{z, f^{-1}(t)}\big| \le 2\mathbf{Lip}\, f \cdot \psi \,. \tag{8.20}$$

9 Persistence of the Poincaré Inequality Under Limits

Let $\{(Z_i, z_i, \rho_i)\}$ be a sequence of length spaces. If $\{(Z_i, z_i, \rho_i)\}$ converges to (Z, z, ρ) in the Gromov-Hausdorff sense, we write $(Z_i, z_i, \rho_i) \xrightarrow{d_{GH}} (Z, z, \rho)$; see [GroLaPa] for the definition and basic facts concerning Gromov-Hausdorff convergence.

Assume in addition that the metrics, ρ_i, are λ-quasi-convex, for some $\lambda < \infty$. Assume in addition, that for the sequence of canonically associated length space metrics, $\rho_{0,i}$, we have $(Z_i, z_i, \rho_{0,i}) \xrightarrow{d_{GH}} (Z, z, \hat{\rho})$ for some to some length space metric, $\hat{\rho}$, on Z. In this case, which can always be obtained by passing to a suitable subsequence, ρ is λ-quasi-convex, $\hat{\rho}$ is λ-quasi-isometric to ρ and $\rho_0 \le \hat{\rho}$.

In the above situation, if μ_i, μ are Borel regular measures on Z_i, Z, we say that $\{(Z_i, z_i, \rho_i, \mu_i)\}$ *converges to* (Z, z, ρ, μ) *in the measured Gromov-Hausdorff sense*, if in addition, for all $\underline{z}_i \to \underline{z}$ and all $r > 0$, we have $\mu_i(B^0_r(\underline{z}_i)) \to \mu(\hat{B}_r(\underline{z}))$, where the balls, $B^0_r(\underline{z}_i), \hat{B}_r(\underline{z})$, are defined with respect to the metrics, $\rho_{0,i}, \hat{\rho}$,

In this case, we write $(Z_i, z_i, \rho_i, \mu_i) \xrightarrow{d_{GH}} (Z, z, \rho, \mu)$.

The above notion of convergence of measures (used in [ChCo2] and somewhat stronger than that of [Fu]) is an appropriate one for our present discussion. The reason for bringing in the metrics $\rho_{0,i}, \hat{\rho}$, stems from Proposition 6.12. To emphasize this point and for the purposes of section 10, we state the following result.

Given a sequence, $\{(Z_i, z_i, d_i, \mu_i)\}$, put $\underline{\mu}_i(\,\cdot\,) = \mu_i(\,\cdot\,)/\mu_i(B_1(z_i))$. The measure, μ in Theorem 9.1 below, is called a *renormalized limit measure.*

Theorem 9.1. *Given a sequence,* $\{(Z_i, z_i, d_i, \mu_i)\}$ *where the* Z_i *are all* λ*-quasi-convex, for some* $\lambda < \infty$*, and* μ_i *satisfies* (0.1)*, with constant,* κ*, independent of* i*, there exists* (Z, z, d, μ)*, with* μ *a Radon measure satisfying* (0.1)*, such that* $(Z_j, z_j, d_j, \underline{\mu}_j) \xrightarrow{d_{GH}} (Z, z, d, \mu)$*, for some sequence,* $\{(Z_j, z_j, d_j, \underline{\mu}_j)\}$.

Proof. As a consequence of Proposition 6.12, this follows from the argument of section 2 of [ChCo2]. □

For the remainder of this section we will assume that (Z,μ) satisfies (0.1), and (4.3), for some $1 \le p < \infty$. Moreover, until the very end (with no essential loss of generality) we assume that Z is a length space.

Recall that if (0.1) holds, and (4.3) holds for some $1 \le p < \infty$, and constant τ_p, then (4.3) also holds for all $p' > p$. The best constant, $\tau_{p'}$, can be bounded above in terms of p, p', κ, τ_p; see section 4.

By using the results of section 5, we will show that for $p' > p$, a strengthened and more stable form of the Poincaré inequality actually holds (with a constant which blows up as $p' \to p$). Also, from the discussion of section 5, it is easy to see that the strengthened form of the inequality passes to Gromov-Hausdorff limits. Thus, on such a limit space, the Poincaré inequality of type $(1,p')$ holds. At this juncture, from the argument given in the proof of Lemma 10.7 (in what is essentially the same context) we can conclude that the constant in the type $(1,p')$ Poincaré inequality on the limit space is bounded below by the lim inf of the best constants in the type $(1,p)$ inequality for the spaces in the approximating sequence.

Fix $1 \le p < \infty$. Let $g \in L_{p'}$ be a generalized upper gradient for $f \in L_{p'}$, with $p' > p$. Then

$$\fint_{B_r(\underline{z})} |f|^{p'}\, d\mu \le 2^{p'} \fint_{B_r(\underline{z})} |f - f_{\underline{z},r}|^{p'} + 2^{p'} |f_{\underline{z},r}|^{p'} . \tag{9.2}$$

For fixed $z_1 \in B_r(\underline{z})$, take $f(z_2) = F_g(z_1,z_2)$, where $F_g(z_1,z_2)$ is as in section 5. Then

$$\fint_{B_r(\underline{z})} F_g(z_1,z_2)^{p'}\, d\mu \le 2^{p'+\kappa} \fint_{B_{2r}(z_1)} \big|F_g(z_1,z_2) - (F_g(z_1,z_2))_{\underline{z},r}\big|^{p'}\, d\mu + 2^{p'+\kappa}\big((F_g(z_1,z_2))_{\underline{z},r}\big)^{p'} . \tag{9.3}$$

If we apply (5.28), integrate with respect to z_1 over $B_{3r}(\underline{z})$ and apply the maximal function inequality for the space, $L_{p'/p}$, we get

$$\left(\fint_{B_r(\underline{z})\times B_r(\underline{z})} F_g(z_1,z_2)^{p'}\, d\mu \times d\mu\right)^{p'} \le c_{p'} r ((g^{p'})_{\underline{z},3r})^{p'} , \tag{9.4}$$

where $c_{p'} = c(\kappa, \tau_{p'}, p)$.

It follows from (5.13), that (9.4) implies the existence of a weak Poincaré inequality of type $(1,p')$, with constant, $c_{p'}$. Moreover, we have:

Theorem 9.5. *Let Z be a length space for which closed balls are compact and let $\{(Z_i, z_i, \mu_i)\}$ converge to (Z,z,μ) in the measured pointed Gromov-Hausdorff sense, where Z_i is a length space for all i. If for $1 \le p < p' < \infty$,* *(0.1), (9.4) hold for all μ_i, with constants, $\kappa, c_{p'}$, independent of i, then (0.1), (9.4) hold for μ, $\kappa, c_{p'}$.*

Proof. By an obvious approximation argument based on Lemma 5.18, the proof can be reduced to the case in which the function, g is *continuous*, (and strictly positive) in which case the assertion is easily verified.

Specifically, from the assumption that Z is a complete length space and g is bounded we find a uniform upper bound on $F_g(z_1, z_2)$, for all $(z_1, z_2) \in B_r(\underline{z})$. Then, since g has a positive lower bound, we conclude that the infimum in the definition of $F_g(z_1, z_2)$ is realized by some rectifiable curve, c, of at most a definite length. This curve can be approximated as well as we like by a rectifiable curve, $c_i \subset Z_i$ (whose length is as close as we like to that of c) provided i is sufficiently large. Similarly, since the function, g is continuous, it can be approximated as well as we like by a function, g_i on Z_i. From this, the proof is easily completed. □

Now we can show the corresponding result for the Poincaré inequality. Note that if (0.1), and (4.3), for some $1 < p < \infty$, hold for all μ_i, with constants, κ, τ_p independent of i, then for any $p' > p$, there exists some *smallest* constant, $\tau_{p'} < \infty$ such that (0.1), (4.3) hold with constants $\kappa, \tau_{p'}$, for all with μ_i; see section 5. The value of $\tau_{p'}$ can be bounded from above in terms of κ, τ_p, p, p'.

Theorem 9.6. *Let Z be a metric space for which closed balls are compact and let $\{(Z_i, z_i, \mu_i)\}$ converge to (Z, z, μ) in the measured pointed Gromov-Hausdorff sense. If (0.1) and (4.3), for some $1 < p < \infty$, hold for all μ_i, with constants, κ, τ_p independent of i, then for $p' > p$, (0.1), (4.3) hold for μ, with constants, $\kappa, \tau_{p'}$.*

Proof. It follows from Theorem 9.5 that (Z, μ), with metric, $\hat{\rho}$ (as defined at the beginning of this section) satisfies a Poincaré inequality of type $(1, p')$ (with some definite constant which might not be optimal). Since the limit metric ρ is λ-quasi-isometric to $\hat{\rho}$, this holds for the metric, ρ, on (Z, μ) as well. However, it follows from Lemma 10.7 of the next section, that if (4.3) holds for *some* constant, then it actually holds with constant $\tau_{p'}$. □

10 Infinitesimal Generalized Linearity

In this section we complete our generalization of Rademacher's theorem on the almost everywhere differentiability of Lipschitz functions on Euclidean spaces, under the assumptions that (Z, μ) satisfies the doubling condition, (0.1) and the Poincaré inequality, (4.3). Specifically, we show that a Lipschitz function, f is infinitesimally generalized linear at $\underline{z}$, for μ-a.e. $\underline{z} \in Z$; see Definition 10.1 and Theorem 10.2.

When specialized to the case of Euclidean space, $\mathbf{R}^n$, Theorem 10.2, together with Theorem 4.38 (which contains the relative uniqueness statement) immediately yield the classical theorem of Rademacher on the almost everywhere differentiability of Lipschitz functions on Euclidean space.

The condition in Definition 10.1 below could be further strengthened by requiring that for a given function, f, and tangent cone, Z_z, any two limit functions differ by composition with an isometry of Z_z. The present formulation corresponds to what we are able to prove (and suffices to recover the classical Rademacher theorem).

We begin with some preliminaries.

The basic facts concerning tangent cones require only (0.1) and the assumption that Z is *proper*, i.e. that closed balls of finite radius are compact. Note that in a metric space which carries a nontrivial doubling measure, closed balls of finite radius are totally bounded. Hence, if complete, such a space is proper.

Recall that a tangent cone, $Z_{\underline{z}}$, is the pointed Gromov-Hausdorff limit, $(Z_{\underline{z}}, \underline{z}_\infty, d_\infty)$, of some sequence, $\{(Z, \underline{z}, r_i^{-1}d)\}$, where $r_i \to 0$. By Gromov's compactness theorem, in the presence of (0.1), tangent cones exist for all $\underline{z} \in Z$ (but need not be unique). From now on, we consider tangent cones together with a choice of renormalized limit measure, i.e. $(Z, \underline{z}, r_i^{-1}d, \underline{\mu}_i) \xrightarrow{d_{GH}} (Z_{\underline{z}}, \underline{z}_\infty, d_\infty, \mu_\infty)$, for some sequence $r_i \to 0$. By Theorem 9.1, such renormalized limit measures exist (but need not be unique). Sometimes we suppress the measure, μ_∞ and just write Z_z.

Let say $d_{GH}((Z, z), (\hat{Z}, \hat{\mu})) < \epsilon$ and fix a Gromov-Hausdorff approximation $\psi : Z \to \hat{Z}$. For $z \in Z, \hat{z} \in \hat{Z}$, we put $\overline{z, \hat{z}} = \overline{\psi z, \hat{z}}$. For continuous functions, $f, \hat{f}$, we put $\overline{f, \hat{f}} = |f - \hat{f} \circ \psi|_{L_\infty}$. Below, this will be understood without mention of the particular choice of Gromov-Hausdorff approximation, ψ.

Definition 10.1. The function, f, is *infinitesimally generalized linear* at $\underline{z} \in Z$, if $\underline{z}$ is a Lebesgue point of $\operatorname{Lip} f$ and if for every tangent cone, $(Z_{\underline{z}}, \mu_\infty)$, every sequence, $r_i \to 0$, such that $(Z, \underline{z}, r_i^{-1}d, \underline{\mu}_i) \xrightarrow{d_{GH}} (Z_{\underline{z}}, \underline{z}_\infty, d_\infty, \mu_\infty)$, and $f_{r_i,\underline{z}} \to f_{0,\underline{z}}$, the function $f_{0,\underline{z}}$ is generalized linear.

Theorem 10.2 (Infinitesimal generalized linearity; Rademacher 3)**.** *Let* (Z, μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Then, for all* $p' > p$, *a Lipschitz function, $f : Z \to \mathbf{R}$, is infinitesimally generalized linear at $\underline{z}$ for μ-a.e. $\underline{z} \in Z$. In particular, at such a point, $f_{0,\underline{z}}$, has the properties set forth in Theorem 8.5. Moreover, $Lip\, f_{0,\underline{z}} \equiv \mathrm{Lip}\, f(\underline{z})$, is independent of the particular function $f_{0,\underline{z}}$ and tangent cone $Z_{\underline{z}}$.*

Proof. By Theorem 6.1, we have $g_f = \mathrm{Lip}\, f$. Moreover, by Theorem 9.6, relation (4.3) holds with some definite constant (which, a priori, might not be optimal) on any tangent cone $(Z_{\underline{z}}, \mu_\infty)$, for all $p' > p$; compare the proof of Theorem 9.6. Thus, the upper gradient energy coincides with the Dirichlet energy on any $(Z_{\underline{z}}, \mu_\infty)$ as well. The fact that we can work with the Dirichlet energy, $|\mathrm{Lip}\, f|_p^p$, plays an essential role throughout our argument. Indeed, the Poincaré inequality only enters the proof indirectly (via Theorem 9.6, via Theorems 5.1, 6.1 and their consequence Corollary 6.36, and finally, via Theorem 4.24).

We must show (see section 8) that for μ-a.e. $\underline{z}$, we have

i) $f_{0,\underline{z}} \equiv 0$ or range$f_{0,\underline{z}} = (-\infty, \infty)$,
ii) $f_{0,\underline{z}}$ is p-harmonic, for all $p' > p$,
iii) $\mathrm{Lip}\, f_{0,\underline{z}} \equiv \mathrm{Lip}\, f(\underline{z})$, for all $Z_{\underline{z}}$ and all $f_{0,\underline{z}}$.

As indicated in the (proof of) Corollary 6.38, we can actually restrict attention to a fixed value $p' \in (p, \infty)$.

By Corollary 6.36, we have for f Lipschitz, $Lip\, f = \mathrm{Lip}\, f$. This will enable us to deduce iii); see Lemma 10.3.

Once ii) has also been established, i) is a direct consequence of Theorem 8.5; see also Corollary 8.6.

The main point is to show ii), which is formally implied by (10.6). Most of our work goes into proving this relation.

The proof of the following Lemma 10.3 is virtually identical to that of Lemma 6.24. Thus, the proof of Lemma 10.3 will be omitted; compare also the proof of Proposition 4.26.

LEMMA 10.3. *Let (Z, μ) satisfy (0.1). Let f be Lipschitz and assume that for μ-a.e. $\underline{z} \in Z$, (6.37) holds. Then for all $\xi, \chi > 0$ and μ-a.e. $\underline{z} \in Z$, there exists $s_f = s_f(\underline{z}, \xi, \chi)$, such that for $0 < r < s_f$ and all $\underline{z}_1, \underline{z}_2, \in B_r(\underline{z})$ with $\overline{\underline{z}_1, \underline{z}_2} \geq \xi r$,*

$$\big|f(\underline{z}_1) - f(\underline{z}_2)\big| < \mathrm{Lip}\, f(\underline{z}) \cdot \overline{\underline{z}_1, \underline{z}_2} + \chi\,. \tag{10.4}$$

Moreover, for all such $\underline{z}_1$, there exists $\underline{z}_2 \in B_{2r}(\underline{z})$ with $\overline{\underline{z}_1, \underline{z}_2} \geq \xi r$, such that

$$\big|f(\underline{z}_1) - f(\underline{z}_2)\big| > \mathrm{Lip}\, f(\underline{z}) \cdot \overline{\underline{z}_1, \underline{z}_2} - \chi\,. \tag{10.5}$$

Let $\underline{z}$ be as in Lemma 10.3 and $\underline{z}_\infty \in Z_{\underline{z}}$, for some $Z_{\underline{z}}$. From the definition of $\operatorname{Lip} f_{0,\underline{z}}(\underline{z}_\infty)$, together with Lemma 10.3, it follows that iii) above holds. In fact, it already follows that for all $\underline{z}_\infty \in Z_{\underline{z}}$, $r > 0$, there exists $\underline{z}'_\infty \in Z_{\underline{z}}$, such that $\overline{\underline{z}_\infty, \underline{z}'_\infty} = r$ and $|f(\underline{z}'_\infty) - f(\underline{z}_\infty)| = \operatorname{Lip} f_{0,\underline{z}} \cdot r$.

To complete the proof, we must show that ii) above holds.

Let $F_\infty : B_R(z_\infty) \to \mathbf{R}$ be a Lipschitz function with the same boundary values as $f_{0,\underline{z}_\infty}$. We can assume without loss of generality that f is asymptotically generalized linear at $\underline{z}$. Under this assumption, we will show

$$\limsup_i \left(\fint_{B_R(\underline{z})} (\operatorname{Lip} f_{r_i,\underline{z}})^p \, d\mu_i \right)^{1/p} \leq \left(\fint_{B_R(z_\infty)} (\operatorname{Lip} F_\infty)^p \, d\underline{\mu}_\infty \right)^{1/p}. \tag{10.6}$$

Claim: The inequality, (10.6), implies ii).

Proof of claim. Note that by iii), it suffices to assume that $\operatorname{Lip} f_{0,\underline{z}} \equiv \operatorname{Lip} f(\underline{z})$. The condition of being asymptotically generalized linear implies in particular that $\underline{z}$ is a Lebesgue point of $\operatorname{Lip} f = \operatorname{Lip} f_{r_j,\underline{z}}$ (which is all that we require for this part of the argument). It follows that the left-hand side of (10.6) is equal to $\operatorname{Lip} f(\underline{z})$. Moreover, we have $\operatorname{Lip} f(\underline{z}) \equiv \operatorname{Lip} f_{0,\underline{z}}$, and the lim sup on the left-hand side of (10.6) can be replaced by lim. In view of Theorem 4.24 (which enables us to restrict attention to the case in which F_∞ is Lipschitz) it follows that (10.6) implies ii).

Contrary to what was needed above, in establishing (10.6), we must (of course) use the assumption that f is asymptotically p-harmonic at $\underline{z}$. The idea is to show that given a comparison function, F_∞, we can construct an approximating sequence of functions on the corresponding sequence of (rescaled) balls, $B_{r_i}(\underline{z})$, such that the Dirichlet energies of these functions converge to the Dirichlet energy of F_∞. Then, the fact that f is asymptotically p-harmonic at $\underline{z}$, yields the desired conclusion.

We need the following general result (and will work in this same degree of generality until the very end of the proof).

From now on, we denote by $\Psi(\epsilon_1, \epsilon_2, \dots, \delta_1, \delta_2, \dots)$, a nonnegative function of $\epsilon_1, \epsilon_2, \dots$ depending *only* on the specified auxiliary parameters, $\delta_1, \delta_2, \dots$, and possibly on additional specified quantities like the function, r_f, such that $\Psi \to 0$, if $\epsilon_i \to 0$ for all i, while the specified auxiliary parameters and quantities remain fixed.

Throughout the remainder of this section, we will assume that all balls of radius 1 centered at the base point have measure 1. This is just a convenient normalization which plays no essential role; compare the definition of renormalized limit measures given in section 9. To simplify the exposition slightly, we will just give the proof of Lemma 10.7 in the case in which Z is a length space. The extension to the general case is essentially trivial; see Remarks 6.4, 6.23.

LEMMA 10.7. *Let (Z,μ) satisfy (0.1), with Z quasi-convex. Let $f : B_R(\bar z)\to \mathbf{R}$ be Lipschitz, with*

$$\mathbf{Lip}\, f \le L|f|_{L_p}\,. \tag{10.8}$$

Let $d_{GH}((Z,\bar z,\mu),(\hat Z,\hat{\bar z},\hat\mu))<\epsilon$ and assume that (0.1) holds for $(\hat Z,\hat{\bar z},\hat\mu)$. Then there is a Lipschitz function $\hat f : B_R(\hat{\bar z})\to\mathbf{R}$, such that for $\Psi = \Psi(\epsilon|R,\kappa,L,r_f)$,

$$\overline{f,\hat f} < \Psi |f|_{L_p}\,, \tag{10.9}$$

$$\big||\hat f|_{L_p} - |f|_{L_p}\big| + \mathbf{Lip}\,\hat f < (L+\Psi)|f|_{L_p}\,, \tag{10.10}$$

$$\left(\int_{B_R(\hat{\bar z})} (\mathrm{Lip}\,\hat f)^p\, d\hat\mu\right)^{1/p} < \left(\left(\int_{B_R(\bar z)} (\mathrm{Lip}\, f)^p\, d\mu\right)^{1/p} + \Psi\right)|f|_{L_p}\,. \tag{10.11}$$

Proof. By scaling, we can assume $|f|_{L_p}=1$. Fix $\eta,\psi,\xi>0$, as in Lemma 6.30 and let $\{B_{r_j}(z_j)\}$, $\{w_{j,k}\}$, $f_\#$, be as in that lemma.

Let $\hat z_j\in\hat Z$, $\hat w_{j,k}\in\hat Z$, satisfy $\overline{z_j,\hat z_j}<\epsilon$, $\overline{w_{j,k},\hat w_{j,k}}<\epsilon$. Thus, $\hat w_{j,k}\in B_{r_j+2\epsilon}(\hat z_j)$.

We now define a function, $\hat f:\hat Z\to\mathbf{R}$ which corresponds to the function, $f_\# : Z\to\mathbf{R}$ of Lemma 6.30. In the trivial case, $(Z,\mu)=(\hat Z,\hat\mu)$, we have $f_\# = \hat f$.

Thus, for all j,k, define the function, $\hat f$, on $\cup_{j,k}\hat w_{j,k}$, by setting $\hat f(\hat w_{j,k}) = f(w_{j,k})$.

Next, for each j, define $\hat f$ to be the canonical extension given by (8.3), of the function, $\hat f$, on $\cup_k \hat w_{j,k}\subset B_{r_j+2\epsilon}(\hat z_j)$.

Finally, define define $\hat f$ on the whole of $\hat Z$ to be the canonical extension given by (8.3), of the function, $\hat f$ on $\cup B_{r_j+2\epsilon}(\hat z_j)\subset\hat Z$.

Since the verification of (10.9)–(10.11) is essentially routine, we will only treat (10.11) in detail.

Note that on each ball, $B_{r_j}(z_j)$, we have $\mathbf{Lip}\, f_\#\le L$. Then by (6.31), we get $\mathbf{Lip}\, f_\# \le L+\Psi(\xi|\kappa,L,\eta)$, on Z. So by (6.32), we have

$$\left(\int_{B_R(\bar z)\setminus\cup_j B_{r_j}(z_j)} (\mathrm{Lip}\, f_\#)^p\, d\mu\right)^{1/p} < \big(L+\Psi(\xi|\kappa,L,\eta)\big)\big(C(\kappa)\eta\big)^{1/p}\,. \tag{10.12}$$

On the other hand, by (6.34), (6.35), we get

$$\left(\int_{\cup_j B_{r_j}(z_j)} (\operatorname{Lip} f_{\#})^p \, d\mu\right)^{1/p} < \left(\int_{\cup_j B_{r_j}(z_j)} (\operatorname{Lip} f)^p \, d\mu\right)^{1/p} + \Psi(\psi,\chi|\kappa,L)\,. \tag{10.13}$$

Similarly, using (6.33), we obtain

$$\left(\int_{B_R(\hat{\bar z})\setminus\cup_j B_{r_j}(\hat z_j)} (\operatorname{Lip} \hat f)^p d\hat\mu\right)^{1/p} < \big(L+\Psi(\xi|\kappa,L,\eta)+\Psi(\epsilon|\kappa,\text{Ł},\eta,\xi,\chi,\psi,r_f)\big) \times \big(C(\kappa)(\eta+\Psi(\epsilon|\kappa,L,\eta,\xi,\chi,\psi,r_f))\big)^{1/p}, \tag{10.14}$$

and

$$\left(\int_{\cup_j B_{r_j}(\hat z_j)} (\operatorname{Lip} \hat f)^p d\hat\mu\right)^{1/p} < \left(\int_{\cup_j B_{r_j}(z_j)} (\operatorname{Lip} f_{\#})^p \, d\mu\right)^{1/p} + \Psi(\epsilon|\kappa,L,\xi,\chi,\psi,r_f)\,. \tag{10.15}$$

Equations (10.12)–(10.15) imply (10.11). To see this, note that if we first fix η, then fix ξ sufficiently small, then fix ψ,χ, sufficiently small and finally, fix ϵ sufficiently small, we can make the term, $\Psi(\epsilon|\kappa,\tau,L,r_f)$, in (10.11) as small as we like. This suffices to complete the proof. □

Let $\{(Z_i,\bar z_i,\mu_i)\}$ be a sequence of measure metric spaces as above and let $\tilde f_i,\tilde F_i : B_R(\bar z_i)\to \mathbf{R}$ be Lipschitz with $\tilde f_i-\tilde F_i \in \overset{\circ}{H}_{1,p}$. We say that $\{\tilde F_i\}$ *minimizes the Dirichlet p-energy asymptotically with respect to the boundary values,* $\{\tilde f_i\}$, if for any sequence, of Lipschitz functions, $F_i : B_R(\bar z_i)\to\mathbf{R}$, with $\tilde f_i - F_i \in \overset{\circ}{H}_{1,p}$, we have

$$\limsup_i |\operatorname{Lip} \tilde F_i|_{L_p} \le \liminf_i |\operatorname{Lip} F_i|_{L_p} \tag{10.16}$$

Corollary 10.17. *Let $(Z_i,\bar z_i,\mu_i)\xrightarrow{d_{GH}}(Z,\bar z,\mu)$ and assume that (0.1) holds for all (Z_i,μ_i), with constant, κ, independent of i. Let $f : B_R(\bar z)\to\mathbf{R}$ be Lipschitz. Let $\tilde f_i,\tilde F_i : B_R(\bar z_i)\to\mathbf{R}$, be sequences of Lipschitz functions such that $\tilde f_i\to f$, and $|\operatorname{Lip}\tilde f_i|_{L_\infty}\le L$, for all i and some L. Assume in addition, that $\{\tilde F_i\}$ minimizes the Dirichlet p-energy asymptotically with respect to the boundary values, $\{\tilde f_i\}$. Then*

$$\limsup_i |\operatorname{Lip}\tilde F_i|_{L_p} \le |\operatorname{Lip} f|_{L_p}\,. \tag{10.18}$$

Proof. For each i, take $(Z_i,\bar z_i,\mu_i)=(\hat Z,\hat z,\hat\mu)$ as in Lemma 10.7, and put $f_i=\hat f$, where $\hat f$ is as in that lemma. It follows from Lemma 10.7 that (10.18) holds with $\tilde F_i$ replaced by f_i.

We can slightly modify the functions, f_i, near $\partial B_R(\bar z_i)$, while leaving the Dirichlet energy unchanged in the limit, $i\to\infty$, so as to obtain a sequence

Lipschitz functions with the same boundary values as the $\tilde{f}_i$; compare the discussion at the beginning of the proof of Lemma 16.19. Thus, (10.18) holds with the $\tilde{F}_i$ replaced by the functions in this sequence. However, since the functions, $\tilde{F}_i$, minimize the p-energy asymptotically, (10.18) holds as well. □

Now let us return to consideration of the functions, $f_{r_i,\underline{z}}$. Recall that the function f is asymptotically p-harmonic at $\underline{z}$. In particular, the sequence, $\{f_{r_i,\underline{z}}\}$, minimizes the Dirichlet p-energy asymptotically with respect to its own boundary values. Thus, Corollary 10.17 implies (10.6). This suffices to complete the proof of Theorem 10.2, our generalization of Rademacher's Theorem. □

Before proceeding further, we point out the following which may help to clarify the discussion.

Let $\{(Z,\underline{z},r_i^{-1}d,\underline{\mu}_i)\} \xrightarrow{d_{GH}} (Z_{\underline{z}},\underline{z}_\infty,d_\infty,\mu_\infty)$ and let $\{\psi_i\}$ be a corresponding sequence of Gromov-Hausdorff approximations. Let $f : Z \to \mathbf{R}$ be Lipschitz. Note that when we speak of $f_{0,\underline{z}}$, in actuality, we must choose a sequence, $\{\psi_i\}$, as above. Otherwise, $Z_{\underline{z}}$ is only well defined up to isometry. However, the collection of all functions, $u : Z_{\underline{z}} \to \mathbf{R}$, such that there exists f and $\{\psi_i\}$ as above, with $u = f_{\underline{z}}$, is invariant under isometries of $Z_{\underline{z}}$.

If we use Theorem 10.2 in place of Theorem 3.7, we obtain the following counterpart of Theorem 4.38.

Theorem 10.19. *There exists $N = N(\kappa,\tau)$ such that the following holds: If (Z,μ) satisfies (0.1), and (4.3), for some $1 < p < \infty$, with Z a length space, then there exist measurable sets, U_α and Lipschitz functions, $f_1,\dots,f_k : Z \to \mathbf{R}$, where $k = k(\alpha) \le N$, such that $\mu(Z \setminus \cup_\alpha U_\alpha) = 0$ and for all $\alpha, f_{(a)}$, the following holds:*

- i) *For all $\underline{z} \in U_\alpha$ and constants, $a_1,\dots,a_k$, the function, $a_1f_1+\cdots+a_kf_k$, is infinitesimally generalized linear at $\underline{z}$. Moreover, for all tangent cones, $Z_{\underline{z}}$, the space spanned by $(f_1)_{0,\underline{z}},\dots,(f_k)_{0,\underline{z}}$ has dimension k.*
- ii) *If $f : Z \to \mathbf{R}$ is Lipschitz, then there exist functions, $b_1^\alpha,\dots,b_k^\alpha$, of class L_∞, such that for μ-a.e. $\underline{z} \in U_\alpha$ and all tangent cones, $Z_{\underline{z}}$, $f_{0,\underline{z}} = b_1^\alpha(\underline{z})(f_1)_{0,\underline{z}} + \cdots + b_k^\alpha(\underline{z})(f_k)_{0,\underline{z}}$.*

11 Small Scale and Infinitesimal Structure

Let (Z,μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Under these assumptions, we show that in a suitable sense, the pair, (Z,μ), exhibits a certain amount of small scale and infinitesimal regularity.

Let $\underline{z} \in Z$ and let $\mathcal{C} = Z_{\underline{z}}$ be a tangent cone at $\underline{z}$. A space, $\mathcal{C}_j$, is an *iterated tangent cone* at $\underline{z} \in Z$, if there exists a sequence, $\mathcal{C}_1, \dots, \mathcal{C}_j$, such that $\mathcal{C}_1 = Z_{\underline{z}}$ is a tangent cone at $\underline{z}$ and for $2 \le i \le j$, $\mathcal{C}_i$ is a tangent cone at some point of $\mathcal{C}_{i-1}$.

Let (W, θ) satisfy (0.1), and (4.3), for some $< p < \infty$.

DEFINITION 11.1. The pair, (W, θ), is *stable* if the maximal dimension, k, of a space $\mathcal{L}^k$, of generalized linear functions on W, is equal to the maximal dimension of a space of generalized linear functions on any tangent cone W_w.

DEFINITION 11.2. A stable pair, (W, θ), is a *generalized Minkowski space*, if every iterated tangent cone, $\mathcal{C}$, is stable.

Note that if (W, θ) is a generalized Minkowski space, then we can choose generalized linear functions, $\ell_1, \dots, \ell_k$, such that i), ii) of Theorem 10.19 hold for *all* $\underline{w} \in W$, with $f_i = \ell_i$.

REMARK 11.3. If the space, W, in Definition 11.2 is a Minkowski space, then the space, $\mathcal{L}^k$, is of course, unique. The possible nonuniqueness allowed in Definition 11.2, arises from the fact that a linear combination of generalized linear functions need not be generalized linear.

EXAMPLE 11.4. The space, $\underline{W}$, consisting of 3 half lines emanating from a single point, together with the measure, $\underline{\theta}$, given by 1 dimensional Hausdorff measure, provides an example of a generalized Minkowski space which is not a Minkowski space. In this case, the dimension of a space of generalized linear functions of maximal dimension is 1, but the space, $\mathcal{L}^1$, is not unique. We point out that $\underline{W}$ cannot arise as a tangent cone of a limit space of a sequence of riemannian manifolds with Ricci curvature bounded below.

REMARK 11.5. The conditions of Definition 11.2 together with Theorem 10.2, might initially suggest that any Lipschitz function, u, can be written as a function of a set of generalized linear functions, $\ell_1, \dots, \ell_k$, which form a basis for a subspace of generalized linear functions, $\mathcal{L}^k$, of maximal dimension, i.e. that such functions are coordinates. However, this need not be the case since the level surfaces (and intersections of level surfaces) of the functions, $\ell_1, \dots, \ell_k$, need not be rectifiably connected; compare Examples 4.62, 11.4 and Conjectures 4.63, 4.65.

Theorem 11.6. *Let* (Z, μ) *satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Let* $\{\mathcal{C}_i\}$ *be a sequence of iterated tangent cones such that such that* $\mathcal{C}_1 = Z_{\underline{z}}$ *is a tangent cone at some* $\underline{z} \in Z$ *and for* $2 \le i \le j$, $\mathcal{C}_i$ *is a tangent cone at* *some point of $\mathcal{C}_{i-1}$. Then there exist $N < \infty$ such that $\mathcal{C}_i$ is a generalized Minkowski space, for all $i \geq N$.*

This follows directly from that fact that there is a definite bound, $N(\kappa, \tau)$, on the dimension of a space of generalized linear functions; see Lemma 4.37.

From Theorem 11.6, together with a standard argument based on the Vitali covering theorem, we obtain:

Theorem 11.7. *Let (Z, μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Then for all $\epsilon > 0$, there exists $r(\epsilon, \kappa, \tau, Z, \mu) > 0$, with the following property. Given $0 < r < r(\epsilon, \kappa, \tau)$, there exists $G_\epsilon = \cup_i B_{r_i}(z_i)$, with $r_i > r(\epsilon, \kappa, \tau, Z, \mu)$, such that $\mu(Z \setminus G_\epsilon) < \epsilon$, and for all $\underline{z} \in G_{\epsilon,r}$, there exists a generalized Minkowski space, (W_i, θ_i) and $w_i \in W_i$, for which, in the measured Gromov-Hausdorff sense,*

$$d_{GH}\big(B_{r_i}(z_i), B_{r_i}(w_i)\big) < \epsilon r_i \,. \tag{11.8}$$

In particular, for μ-a.e. $\underline{z} \in Z$, there exists a tangent cone, $Z_{\underline{z}}$, which is a generalized Minkowski space.

Remark 11.9. One would like to obtain the stronger result that at almost all points of Z, every tangent cone is a generalized Minkowski space. Given the stronger assumptions of sections 15, 16, for this it would suffice to have an a priori estimate for $|\mathrm{Lip}\,(\mathrm{Lip}\, h)|_{L_p}$, for some $p \geq 1$, where h is a p-harmonic function. Then Lemma 16.39 and Proposition 16.43 together with the arguments of section 2 of [ChCo2] could be applied. However, in so far as we are aware, the issue of obtaining such estimates is not well understood even for smooth Finsler manifolds (in which context, it is closely related to the existence of a suitable Bochner formula).

12 Norms on T^*Z and Length Space Metrics

Let (Z, μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Let ρ denote the underlying metric on Z. Let $|\cdot|$ denote the canonical pointwise norm on the cotangent bundle T^*Z, i.e. for f Lipschitz, $|df(\underline{z})| = g_f(\underline{z}) = \mathrm{Lip}\, f(\underline{z})$, for μ-a.e. $\underline{z} \in Z$.

Let $||\cdot||$, denote an equivalent norm on T^*Z, i.e. for some constant, $0 < \lambda < \infty$, we have $\lambda^{-1}|\psi(z)| \leq ||\psi(z)|| \leq \lambda|\psi(z)|$; see the discussion prior to Theorem 4.53. We will associate to $||\cdot||$, a length space metric, $\overline{\rho}$, which is $c(\kappa, \tau)$-bi-Lipschitz equivalent to ρ, and show that the canonical norm, $|||df(z)||| = \mathrm{Lip}_{\overline{\rho}}\, f(z)$, associated to $\overline{\rho}$ satisfies $||\cdot|| \leq |||\cdot|||$, μ-a.e. Here

$\mathrm{Lip}_{\overline{\rho}} f(z)$ denotes the pointwise Lipschitz constant defined with respect to the metric $\overline{\rho}$. We will then give a sufficient condition for the equality of the norms, $||df(\underline{z})|| = |||df(\underline{z})||| = \mathrm{Lip}_{\overline{\rho}} f(\underline{z})$, for $\underline{z}$ lying in a given measurable subset.

For the application in section 13, the crucial point is that properties of $|| \cdot ||$, notably strict convexity of the induced norm on $TZ_{\underline{z}} := (T^*Z_{\underline{z}})^*$, can be specified in advance, while in addition, for all $f_{0,\underline{z}}$ lying in a space of blown up limit functions on a tangent cone, $Z_{\underline{z}}$, we have $\mathrm{Lip}_{\overline{\rho}} f(\underline{z}) = \sup_{w' \in \partial B_r(w)} (f_{0,\underline{z}}(w') - f_{0,\underline{z}}(w))/r$, for all $B_r(w) \subset Z_{\underline{z}}$. The fact that these two properties hold simultaneously will enable us to deduce that the canonical Lipschitz map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$, is surjective; see section 13.

Fix a collection, $\{U_\alpha\}$ as in Theorem 4.38. Specifically, we assume that for all α, the norm, $|| \cdot ||$, is well defined on T^*Z_z, for all $z \in U_\alpha$.

Fix $\underline{z} \in Z$. Since the norm, $|| \cdot ||$, is λ-quasi-isometric to $| \cdot |$, μ-a.e. there is a nonempty set of Lipschitz functions, h, satisfying

$$h(\underline{z}) = 0\,, \tag{12.1}$$

$$\big\|dh(z)\big\| \le 1 \qquad\qquad (\text{for } \mu\text{-a.e. } z)\,. \tag{12.2}$$

Since Z is $c(\kappa, \tau)$-quasi-convex, it follows easily from Theorem 4.14 that $|h|$ is bounded above by $c(\kappa, \tau, \lambda)\overline{\underline{z}, z}$ and that the Lipschitz constant of $|h|$ with respect to the given metric is bounded by a definite constant $c(\kappa, \tau, \lambda)$.

Put

$$\overline{\rho}_{\underline{z}}(z) = \sup_h h(z)\,, \tag{12.3}$$

where the sup is taken over all Lipschitz function, h, satisfying (12.1), (12.2).

Clearly, $\overline{\rho}_{\underline{z}}$ satisfies (12.1). Also, it is obvious that $c(\kappa, \tau, \lambda)\overline{\underline{z}, z} \le \overline{\rho}_{\underline{z}}(z)$, since the Lipschitz function, $c(\kappa, \tau, \lambda)\overline{\underline{z}, z}$, satisfies, (12.1), (12.2).

We claim that $\overline{\rho}_{\underline{z}}$ is Lipschitz and satisfies (12.2) as well. To see this, note that if h_1, h_2 are Lipschitz functions, then at almost all points of the set on which the function, $\max(h_1, h_2)$ is equal to h_i, the differential of $\max(h_1, h_2)$ is equal to dh_i. Moreover, the Lipschitz constant of the function, $\max(h_1, h_2)$, is at most $\max(\mathbf{Lip}\, h_1, \mathbf{Lip}\, h_2)$. Thus, if h_1, h_2 satisfy (12.1), (12.2) and have a definite bound on their Lipschitz constants, then the same holds for $\max(h_1, h_2)$.

Choose a countable dense set of points, $\{z_i\}$, and for each i, j, a function, $h_{i,j}$, such that $h_{i,j}(z_i) \ge \overline{\rho}_{\underline{z}}(z_i) - j^{-1}$. If we put $h_j = \max(h_{1,j} \ldots, h_{j,j})$, the function, h_j, satisfies (12.1), (12.2) and is Lipschitz. Moreover, the sequence of functions, $\{h_j\}$ converges uniformly on compact sets to the function $\overline{\rho}_{\underline{z}}$. Clearly, $\overline{\rho}_{\underline{z}}$ is Lipschitz, with a definite bound on its Lipschitz constant.

Now restrict attention to a ball, $B_R(\overline{z})$, with $0 < R < \infty$. From Theorem 4.53, we get $||d\overline{\rho}_{\underline{z}}(z)|| \leq 1$, for μ-a.e. $z \in B_R(\overline{z})$. Hence, by letting $R \to \infty$, it follows that the function, $\overline{\rho}_{\underline{z}}$, satisfies (12.2).

Put $\overline{\rho}(z_1, z_2) = \overline{\rho}_{z_1}(z_2)$. It is trivial to verify that $\overline{\rho}$ is a metric on Z, which is bi-Lipschitz to the original one. Hence, (0.1), (4.3) hold for the metric, $\overline{\rho}$, with definite constants.

PROPOSITION 12.4. *The metric, $\overline{\rho}$, is a length space metric.*

Proof. Put $\overline{\rho}_{\underline{z},\epsilon}(z) = \inf_{\{z_i\}} \sum_0^{N-1} \overline{\rho}(z_i, z_{i+1})$, where the inf is take over all sequences, $\{z_i\}$, with $z_0 = \underline{z}$, $z_N = z$, such that $\overline{\rho}_{z_i}(z_{i+1}) < \epsilon$, for all $0 \leq i \leq N-1$; compare section 17.

Since Z is complete, it suffices to show that $\overline{\rho}_{\underline{z},\epsilon} = \overline{\rho}$.

By an argument similar to the one showing that the function, $\overline{\rho}_{\underline{z}}$, satisfies (12.1), (12.2) and is Lipschitz, but employing in addition, induction on the length of the sequence, $\{z_i\}$, it follows that the function, $\overline{\rho}_{\underline{z},\epsilon}$, also satisfies (12.1), (12.2) and is Lipschitz. Since it is clear that $\overline{\rho}_{\underline{z}} \leq \overline{\rho}_{\underline{z},\epsilon}$, it follows from the definition of $\overline{\rho}_{\underline{z}}$, that in fact, $\overline{\rho}_{\underline{z}} = \overline{\rho}_{\underline{z},\epsilon}$. This suffices to complete the proof. □

Let $|||\cdot|||$ denote the norm on T^*Z associated to the metric $\overline{\rho}$. We have $\operatorname{Lip}_{\overline{\rho}} \overline{\rho}(\underline{z}, z) = 1$, for all z, and hence, $|||d\overline{\rho}(\underline{z}, z)||| = 1$, for μ-a.e. z. On the other hand, by construction, $||d\overline{\rho}(\underline{z}, z)|| \leq 1$, for μ-a.e. $z \in Z$. Thus, $||d\overline{\rho}(\underline{z}, z)|| \leq |||d\rho(\underline{z}, z)||$, for μ-a.e. z.

Theorem 12.5. *For all α, there exists $V_\alpha \subset U_\alpha$, with $\mu(U_\alpha \setminus V_\alpha) = 0$, such that on V_α, we have $||\cdot|| \leq |||\cdot|||$.*

Proof. Fix a ball, $B_R(\overline{z})$, with $0 < R < \infty$. Let f_i be a sequence of functions as in Theorem 6.5, converging uniformly to f. Just as in the proof of Proposition 6.2, by applying Theorem 4.53, we get $||df(\underline{z})|| \leq |||df(\underline{z})|||$, for μ-a.e. $\underline{z}$.

Let the notation, $f^\alpha_{(a)}$, be as in Theorem 4.38. It follows from the above that there exists a subset, $V_\alpha \subset U_\alpha$, with $\mu(U_\alpha \setminus V_\alpha) = 0$, such that $||df^\alpha_{(a)}(\underline{z})|| \leq |||df^\alpha_{(a)}(\underline{z})|||$, for μ-a.e. $\underline{z} \in U_\alpha$ and all *rational* (a). As in Lemma 4.35, it follows that the same holds for (a) arbitrary. □

REMARK 12.6. Let $\operatorname{Lip}_\rho f(z)$ denote the pointwise Lipschitz constant defined with respect to the metric, ρ, on Z. If we take $||\cdot||$, such that $||df(z)|| = \operatorname{Lip}_\rho f(z)$, it follows from Proposition 6.2, that $\rho \leq \overline{\rho}$. In Corollary 15.14, we will give a condition which implies $\rho_0 = \overline{\rho}$.

The following theorem provides a sufficient condition, (12.8), which ensures that on some fixed $U_{\underline{\alpha}}$, we have norm, $||\cdot|| = |||\cdot|||$, for μ-a.e. $\underline{z} \in U_{\underline{\alpha}}$. (Thus, if (12.8) holds for all $\underline{\alpha}$, then $||\cdot|| = |||\cdot|||$ μ-a.e.)

Theorem 12.7. *Assume that for some $\underline{\alpha}$, all $\underline{z} \in U_{\underline{\alpha}}$ and all $v \in T^*Z_{\underline{z}}$, there exists a Lipschitz function, f_v, with $df_v(\underline{z}) = v$, such that*

$$||df_v(z)|| \leq ||df_v(\underline{z})|| \qquad \text{(for } \mu\text{-a.e. } z \in Z\text{)}\,. \tag{12.8}$$

Then there exists $V_{\underline{\alpha}} \subset U_{\underline{\alpha}}$, with $\mu(U_{\underline{\alpha}} \setminus V_{\underline{\alpha}}) = 0$, such that on $V_{\underline{\alpha}}$, we have $||\cdot|| = |||\cdot|||$.

Proof. Let $v = df_v(\underline{z})$ be as above. Since $\rho(\underline{z}, z)$ is the largest Lipschitz function satisfying (12.1), (12.2), it follows that $|f_v(z) - f_v(\underline{z})| \leq ||df_v(\underline{z})||\rho(\underline{z}, z)$. Hence, for all $\underline{z} \in U_{\underline{\alpha}}$ and $v \in T^*Z_{\underline{z}}$, we get $\mathrm{Lip}_{\overline{\rho}}\, f_v(\underline{z}) \leq ||df_v(\underline{z})||$, or equivalently, $|||v||| \leq ||v||$. In view of Theorem 12.5, this suffices to complete the proof. □

13 Hausdorff Measure

Let (Z, μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$, with Z complete. From now on, we fix a collection, $\{U_\alpha\}$, as in Theorem 4.38. In this section, with the aid of Theorem 12.7, we prove certain weakened versions of Conjecture 4.63 For instance, we show that for μ-a.e. $\underline{z} \in U_\alpha$, we have $\mathcal{H}^{k(\alpha)}(B_r(w)) \geq c(\kappa, \tau) r^{k(\alpha)}$, for all tangent cones $Z_{\underline{z}}$ and balls, $B_r(w) \subset Z_{\underline{z}}$. Moreover, there is natural surjective Lipschitz map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$; see also Theorems 13.8, 13.12.

We will begin by making some simple observations which imply in particular, that if $\underline{z} \in U_\alpha$ and the norm, $|df(\underline{z})| = \mathrm{Lip}\, f(\underline{z})$, on $T^*Z_{\underline{z}}$ is *strictly convex*, then $\dim(Z_{\underline{z}}) \geq k(\alpha)$, for *every* tangent cone $Z_{\underline{z}}$. Although this condition need not hold for the given metric, using the results of section 12, we can modify the metric so as to achieve it, at least on a given $U_{\underline{\alpha}}$, for arbitrary $\underline{\alpha}$. This suffices for our purposes.

Let us recall some standard facts concerning finite dimensional normed linear spaces. The following discussion, although more general than is actually required, may serve to put matters in perspective.

Let V be a finite dimensional vector space with dual space, V^*. For $0 \neq v \in V$, put $D_v = \{v^* \in V^* \,|\, v^*(v) = 1\}$. Let T_v denote the set of hyperplanes, H, such that $v \notin H$. For $v^* \in V^*$, let $N(v^*)$ denote the null space of v^*. The map, $A : D_v \to T_v$, defined by $A(v^*) = N(v^*)$, is a bijection.

Now assume that V carries a norm, $|\cdot|$, and let $S(V)$ denote the unit sphere. Let V^* carry its canonical norm, $|v^*| = \sup_{v\in S(V)} v^*(v)$. Since the unit ball in V is convex, for all $v \in S(v)$, there is at least one *supporting hyperplane*. Of necessity, $H \in T_v$.

Let $R(V)$, the *regular set*, denote the set of points, $v \in S(V)$, such that the supporting hyperplane, H is unique. Let $W(V)$, the *weakly strictly convex set*, denote the set of points, $v \in S(V)$, such that there exists a supporting hyperplane, H, (where of necessity, $H \in T_v$) such that $H \cap S(V) = v$. Clearly, the sets, $R(V)$, $S(V)$, are invariant under the antipodal map.

The norm function on V is Lipschitz and hence differentiable almost everywhere. Since this function is also homogeneous of degree 1, it follows that $v \in R(V)$, for almost all $v \in S(V)$. If $v \in R(V)$, then H has *first order contact* with $S(V)$ at v. Conversely, if there exists a hyperplane having first order contact with $S(V)$ at v, then $v \in R(V)$ and this hyperplane is supporting.

If $|v| = 1$ and $v^* \in D_v$, then $|v^*| = 1$, if and only if $N(v^*)$ is a supporting hyperplane at v. Conversely, if $|v^*| = 1$, then there exists $v \in S(V)$ such that $N(v^*)$ is a supporting hyperplane at v.

Define the *Legendre transformation*, $L : R(V) \to S(V^*)$, by $L(v) = A^{-1}(H)$, where H is the unique supporting hyperplane at v. It follows easily that $L(R(V)) = W(V^*)$, i.e. the image of the regular set under L coincides with the set of points of weak strict convexity of $S(V^*)$. Although in general, the set, $W(V^*)$, can have measure zero in $S(V^*)$, we have $W(V^*) = V^*$ if the norms in question are strictly convex, e.g. if they come from inner products. Indeed, this case will suffice for our purposes.

Let X denote a compact topological space. Let $\mathcal{V}$ denote a finite dimensional vector space of bounded functions on X. Equip $\mathcal{V}$ with the sup norm. Let $e : X \to \mathcal{V}^*$ denote the evaluation map, $e(v) = v(x)$.

LEMMA 13.1. *Let $v^* \in W(\mathcal{V}^*)$, with $L(v) = v^*$, $v \in R(V)$. If there exists $x \in X$ such that $|v| = v(x) = 1$, then $v^* \in$ range(e). In particular, for all $v^* \in W(\mathcal{V}^*)$, either $v^* \in$ range(e) or $-v^* \in$ range(e)*

Proof. Let $L(v) = v^*$, with $v \in R(V)$. Let x be such that $|v| = v(x) = 1$.

It suffices to show that $e(x) = v^*$. To see this, we must verify that for all $h \in H = N(v^*)$, we have $h(x) = 0$. After replacing h by $-h$ if necessary, we can assume $h(x) \geq 0$. Then for $t > 0$, we have $1 \leq (v + th)(x) \leq |v + th| = 1 + o(t)$, where the last equality follows from the fact that H has first order contact with $S(V)$ at v. Thus, $h(x) = 0$. $\square$

As usual, let dim denote Hausdorff dimension and let $\mathcal{H}^k$ denote k-dimensional Hausdorff measure.

COROLLARY 13.2. *Let X be a compact metric space. Let $\mathcal{V}$ denote a k-dimensional space of Lipschitz functions on X. If the sup norm on $\mathcal{V}$ has the property that* $\dim(W(\mathcal{V}^*)) = k-1$, *then* $\dim(X) \geq k-1$. *If in addition, $\mathcal{H}^{k-1}(W(V^*)) > 0$, then $\mathcal{H}^{k-1}(X) > 0$.*

Let $\mathcal{L}^{k(\alpha)}$ denote a space of dimension, $k(\alpha)$, of generalized linear limit functions on some tangent cone, $Z_{\underline{z}}$, with $\underline{z} \in U_\alpha$; see Theorem 10.19. If f is infinitesimally generalized linear at $\underline{z}$, and $f_{0,\underline{z}} \in \mathcal{L}^{k(\alpha)}$, then we have $|df(\underline{z})| = \mathrm{Lip}\, f(\underline{z}) = \sup_{w' \in \partial B_r(w)} (f_{0,\underline{z}}(w') - f_{0,\underline{z}}(w))/r$, for all $B_r(w) \subset Z_{\underline{z}}$; see Theorem 8.6.

Recall that by definition, the *tangent space*, $TZ_{\underline{z}}$, at $\underline{z}$ is the vector space, $(T^*Z_{\underline{z}})^*$. Associated to the space, $\mathcal{L}$, of the previous paragraph, is the map $e : Z_{\underline{z}} \to TZ_{\underline{z}}$.

As above, we get:

COROLLARY 13.3. *Let (Z, μ) satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$, *with Z a complete length space. If $\underline{z} \in U_\alpha$ and the canonical norm on $T^*Z_{\underline{z}}$ satisfies $W(\mathcal{L}^*) = S(\mathcal{V}^*)$, then the map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$, is surjective and for all $B_r(w) \subset Z_{\underline{z}}$, we have $\mathcal{H}^{k(\alpha)}(B_r(w)) \geq c(k(\alpha))\, Vol(B_r(0))$.*

The following strengthening of Corollary 13.3 is the main result of this section.

Theorem 13.4. *Let (Z, μ) satisfy* (0.1), *and* (4.3), *for some* $1 < p < \infty$. *Then for all U_α, there exists $V_\alpha \subset U_\alpha$, with $\mu(U_\alpha \setminus V_\alpha) = 0$, such that for all $z \in V_\alpha$, every tangent cone, $Z_{\underline{z}}$ and ball, $B_r(w) \subset Z_{\underline{z}}$,*

$$\mathcal{H}^{k(\alpha)}(B_r(w)) \geq c(k(\alpha))\, Vol(B_r(0))\,, \tag{13.5}$$

where $B_r(0) \subset \mathbf{R}^{k(\alpha)}$. Moreover, the map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$, is surjective. In particular, $\dim(Z_{\underline{z}}) \geq k(\alpha)$.

Proof. Note that if the metrics ρ_1, ρ_2 satisfy (0.1), (4.3) and are bi-Lipschitz equivalent, then for μ-a.e. $\underline{z}$ and all tangent cones, $Z_{\underline{z}}$, a space of generalized linear limit functions, $f_{0,\underline{z}}$, with respect to ρ_1, will also be such a space with respect to ρ_2. Hence, at such points, the map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$, is independent of the particular metric, ρ_1, ρ_2.

Fix $\alpha = \underline{\alpha}$. In view of Corollary 13.3, together with what was noted in the previous paragraph, it suffices to show that for all $U_{\underline{\alpha}}$, their exists a length space metric, bi-Lipschitz equivalent to the given one, such that when resticted to $V_{\underline{\alpha}}$ as above, the associated norm, $|||\cdot|||$, on T^*Z, is strictly convex.

For this, it suffices to exhibit a corresponding norm, $||\cdot||$, on T^*Z, for which the hypothesis, (12.8), of Theorem 12.7, is satisfied.

Let the notation be as Theorem 4.38. From the discussion preceding Theorem 4.48, together with a straightforward argument based on Lusin's theorem, it follows that we can assume that for all α and all $\underline{z}\in U_\alpha$,

$$c(k)^{-1}\big(|a_1|^2+\cdots+|a_{k(\alpha)}|^2\big)^{\frac{1}{2}} \le \big|d^{\underline{\alpha}}_{(a)}(\underline{z})\big| \le c(k)\big(|a_1|^2+\cdots+|a_{k(\alpha)}|^2\big)^{\frac{1}{2}}. \tag{13.6}$$

Clearly, we can assume that there exists, $L<\infty$, such that $|df^{\underline{\alpha}}_{(a)}(z)| \le L|df^{\underline{\alpha}}_{(a)}(z)|$, for all (a) and μ-a.e. $z\in Z$. Define the norm, $||\cdot||$, by putting $||df^{\underline{\alpha}}_{(a)}(z)|| = (|a_1|^2+\cdots+|a_{k(\underline{\alpha})}|^2)^{\frac{1}{2}}$, for all $z\in U_{\underline{\alpha}}$, and $||df(z)|| = L^{-1}|df(z)|$ for all $z\in U_\alpha$ and all $\alpha\neq\underline{\alpha}$. It is obvious that for this $||\cdot||$, (12.8) holds. ◻

We close this section with two results which provide somewhat weakened versions Conjecture 4.63.

Recall that the measure, μ, is called *Ahlfors k-regular* if there exist constants, $0<c_1\le c_2<\infty$, such that for all $\underline{z}\in Z$ and $0<r<\infty$, we have

$$c_1 r^k \le \mu(B_r(\underline{z})) \le c_2 r^k\,. \tag{13.7}$$

Theorem 13.8. *Let (Z,μ) satisfy (0.1), and (4.3), for some $1<p<\infty$. Assume that there exist $c, r'>0$, such that for all $\underline{z}\in Z$ and all $0<r<r'$,*

$$cr^k \le \mu(B_r(\underline{z}))\,. \tag{13.9}$$

Then for all α,

$$k(\alpha)\le \dim(Z_{\underline{z}}) \le k\,. \tag{13.10}$$

If in addition, (13.5) holds, for all $\underline{z}\in Z$ and all $0<r<r'$, then for all α,

$$k(\alpha)\le \dim(U_\alpha)=\dim(Z_{\underline{z}})=k\,. \tag{13.11}$$

Proof. It is clear that the validity of either (13.7) or (13.9) for all $\underline{z}\in Z$ and all $0<r\le r'$, implies that that the corresponding relation holds for every tangent cone, $Z_{\underline{z}}$, equipped with any renormalized limit measure. By a standard argument, (13.9) implies $\dim(Z_{\underline{z}})\le k$, while (13.7) implies $\dim(U_\alpha)=k$, $\dim(Z_{\underline{z}})=k$. From this together with Theorem 13.4, our assertions follow. ◻

Let the map, f^α, be as in Conjecture 4.63. According to that conjecture, we have $\mathcal{H}^{k(\alpha)}(f^\alpha(U_\alpha))>0$.

Theorem 13.12. *Let (Z,μ) satisfy (0.1), and (4.3), for some $1<p<\infty$. Let $K_\alpha\subset U_\alpha$, with $\mu(K_\alpha)>0$. If (13.9) holds for all $\underline{z}\in K_\alpha$, with $k=k(\alpha)$, then $\mathcal{H}^{k(\alpha)}(f^\alpha(K_\alpha))>0$.*

Proof. By Theorem 13.4, we can assume that the map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$, is surjective, for all $\underline{z} \in K_\alpha$ and all tangent cones $Z_{\underline{z}}$. From this and a straightforward modification of the proof of Theorem 0.1 of [Co], the conclusion follows. We omit the details. □

14 Subsets of $\mathbf{R}^N$ and Bi-Lipschitz Nonimbedding

Let (Z, μ) satisfy (0.1), and (4.3), for some $1 < p < \infty$. Fix a collection, $\{U_\alpha\}$ as in Theorem 4.38.

In this section we show the following: Assume that there exists an isometric imbedding $i : Z \to \mathbf{R}^N$, for some N, i.e. $d(i(z_1), i(z_2)) = \overline{z_1, z_2}$, where d denotes distance in $\mathbf{R}^N$. Then for all α, there exists $V_\alpha \subset U_\alpha$, with $\mu(U_\alpha \setminus V_\alpha) = 0$, such that for all $\underline{z} \in V_\alpha$, the tangent cone, $i(Z)_{i(\underline{z})} \subset \mathbf{R}^N_{i(\underline{z})}$, is *unique in the strong sense*; see below for the definition. Moreover, the tangent cone, $i(Z)_{i(\underline{z})} \subset \mathbf{R}^N_{i(\underline{z})}$, is actually a *subspace of dimension* $k(\alpha)$. For every tangent cone, $Z_{\underline{z}}$, there is an (essentially canonical) isometry of $Z_{\underline{z}}$ with $i(Z)_{i(\underline{z})}$. By a standard density lemma, this implies $\dim(V_\alpha) \le k(\alpha)$. If $\mathcal{H}^{k(\alpha))}(V_\alpha) > 0$, then V_α is $k(\alpha)$-rectifiable. According to Conjecture 4.63, this always holds.

If we suppose more generally, that there exists a bi-Lipschitz imbedding, $F : Z \to \mathbf{R}^N$, then for all $\underline{z} \in V_\alpha$, every tangent cone, $Z_{\underline{z}}$, is bi-Lipschitz equivalent to $\mathbf{R}^{k(\alpha)}$ and the additional conclusions above hold without further modification.

By an observation of Semmes, a (nonflat) Carnot-Caratheodory space does not admit a bi-Lipschitz imbedding into $\mathbf{R}^N$, for any N; see [Se4], where this result is attributed independently to Assouad (unpublished). The proof is a simple consequence of Pansu's differentiability theorem for Lipschitz maps between Carnot-Caratheodory spaces; see [Pa]. For the spaces constructed in [L], a corresponding nonimbedding theorem has recently been proved by Laakso, via a direct argument. Our general result implies those of Semmes, Assouad and Laakso, and enables one to treat the spaces considered in [BoP] as well.

The results of this section are obtained as an application of Theorems 4.38, 12.7, 13.4. We are indebted to Stephen Semmes for suggesting that some sort of general bi-Lipschitz nonimbedding theorem could hold.

We begin by collecting some basic facts concerning tangent cones of subsets of $\mathbf{R}^N$ and induced maps on tangent cones in general.

Recall that the existence of a canonical family of homotheties fixing any point, $x \in \mathbf{R}^N$, enables us to define the tangent cone, $\mathbf{R}^N_x$, unambiguously, i.e. any two tangent cones are canonically identified and these identifications are mutually consistent. We express this by saying that for all $x \in \mathbf{R}^N$, the tangent cone, $\mathbf{R}^N_x$, is *unique in the strong sense*. Of course this tangent cone can be canonically identified with the tangent space to $\mathbf{R}^N$ at x.

More generally, let $A \subset \mathbf{R}^N$, be equipped with the induced metric. If, under rescaling at x, the set, A, converges in the Gromov-Hausdorff sense, to a unique subset of $\mathbf{R}^N_x$, we say that *the tangent cone, $A_x \subset \mathbf{R}^N_x$, is unique in the strong sense.*

If we just consider A qua metric space, then tangent cones, A_x, are only well defined up to isometry. Even if any two such intrinsic tangent cones, A'_x, A''_x are isometric, the tangent cone $A_x \subset \mathbf{R}^N_x$, may fail to be unique in the strong sense. However, if $A_x \subset \mathbf{R}^N$ is unique in the strong sense, it is certainly isometric to any intrinsically defined tangent cone.

More generally, let Z_j be proper and assume that μ_j satisfies (0.1) $j = 1, 2$. Let $F : Z_1 \to Z_2$ be Lipschitz. Given, a tangent cone, $(Z_1)_{\underline{z}}$, by passing to a suitable subsequence $\{r_k\}$ of $\{r_i\}$, we obtain a tangent cone, $(Z_2)_{F(\underline{z})}$, and an induced map $F_* : Z_{\underline{z}} \to Z_{F(\underline{z})}$, which depends only on the subsequence (and is otherwise canonical). Similarly, given $(Z_2)_{F(\underline{z})}$, by passing to a subsequence, we obtain, $(Z_1)_{\underline{z}}$, and a map, $F_* : Z_{\underline{z}} \to Z_{F(\underline{z})}$.

Let $\mathcal{V}$ denote a finite dimensional space of Lipschitz functions on Z_2. For all $\underline{z} \in Z_1$, there exists a subsequence, $\{r_\ell\}$ of $\{r_k\}$, such that for all $f \in \mathcal{V}$, the limit function, $f_{0,F(\underline{z})}$, exists. Let $\mathcal{V}_{0,\underline{z}}$ denote the space spanned by the functions, $f_{0,F(\underline{z})}$, where $f \in \mathcal{V}$. The canonical map, $f \to f_{0,F(\underline{z})}$, is linear.

Let $F^{\underline{*}}$ denote the adjoint of the map F_*, i.e. the map on functions induced by composition with F_*. (Recall that in section 4, we defined the map, $F^* : T^*Z_2 \to T^*Z_1$.) It is straightforward to check that the limit space, $(\mathcal{V} \circ F)_{0,\underline{z}}$, exists for the rescaling sequence $\{r_\ell\}$. Moreover, we have $F^{\underline{*}}(\mathcal{V}_{0,F(\underline{z})}) = (\mathcal{V} \circ F)_{0,\underline{z}}$ and the map, $F^{\underline{*}}$, is given by $F^{\underline{*}}(f_{0,F(\underline{z})}) = (f \circ F)_{0,\underline{z}}$.

Theorem 14.1. *Let (Z, μ) satisfy* (0.1), *and* (4.3), *for some $1 < p < \infty$, with Z complete. Assume in addition, that there exists an isometric imbedding, $i : Z \to \mathbf{R}^N$, for some $N < \infty$. Then for all sets, U_α, as in Theorem 4.38, there exists a subset, V_α, with $\mu(U_\alpha \setminus V_\alpha) = 0$, such that for all $\underline{z} \in V_\alpha$, the tangent cone, $i(Z)_{i(\underline{z})} \subset \mathbf{R}^N$, is unique in the strong sense and equal to a subspace of dimension $k(\alpha)$. Moreover,* $\dim(V_\alpha) \le k(\alpha)$.

Proof. Let $i : Z \to \mathbf{R}^N$ denote an isometric imbedding.

Let $\mathcal{V}$ denote the space of Lipschitz functions on $\mathbf{R}^N$ spanned by the coordinate functions, $x_1, \ldots, x_N$. As in the proof of Theorem 4.38, for all α, there exists a subset, $V_\alpha \subset U_\alpha$, such that for all constants, $a_1, \ldots, a_N$, the function, $a_1(x \circ i)_1 + \cdots + a_N(x_N \circ i)$, is asymptotically generalized linear, for all $\underline{z} \in V_\alpha$. For any choice of i_* (gotten, as above, by choosing a suitable sequence of rescalings) we have, $i^{\underline{*}}(\mathcal{V}_{0,i(\underline{z})}) = (\mathcal{V} \circ i)_{0,\underline{z}}$. Thus, it follows that the space, $i^{\underline{*}}(\mathcal{V}_{0,i(\underline{z})})$, consists of generalized linear functions, which each of which is the blown up limit of a Lipschitz function, which is asymptotically generalized linear at $\underline{z}$. Hence, $\dim i^{\underline{*}}(\mathcal{V}_{0,i(\underline{z})}) \leq k(\alpha)$.

Let $T_{i(\underline{z})}$, denote the common null space of the space of linear functions, $a_1x_1 + \cdots + a_N x_N$, whose differentials are in the kernel $N(i^*)_{i(\underline{z})}$, of the map $i^* : T^*\mathbf{R}^N \to T^*Z_{i(\underline{z})}$. It follows that for any map, i_*, as above, we have $i_*(Z_{\underline{z}}) = i(Z)_{i(\underline{z})} \subset T_{i(\underline{z})}$. Note that $\dim(T_{i(\underline{z})}) = \dim i^{\underline{*}}(\mathcal{V}_{0,i(\underline{z})}) \leq k(\alpha)$.

Since i is an isometric imbedding, i_* is an isometric imbedding as well. Hence, we get $\dim Z_{\underline{z}} \leq k(\alpha)$, for all $\underline{z}$ as above and any tangent cone, $Z_{\underline{z}}$. By a standard density lemma for Hausdorff measure (see, e.g. [F]) it follows that $\dim(V_\alpha) \leq k(\alpha)$.

By Theorem 13.4, after replacing V_α by a suitable subset of full measure, we can assume that the map, $e : Z_{\underline{z}} \to TZ_{\underline{z}}$, is surjective, for all $\underline{z} \in V_\alpha$ and all tangent cones, $Z_{\underline{z}}$. On the other hand, if we identify the space, $T_{i(\underline{z})}$, with its double dual, then the natural map, $i_{\underline{*}} : TZ_{\underline{z}} \to T_{i(\underline{z})}$, is clearly surjective.

Since, as is easy to check, $i_{\underline{*}} \circ e = i_*$, it follows that this map is both an isometric imbedding and surjective. This suffices to complete the proof. □

By modifying the definition of V_α if necessary, we can assume that $\underline{z}$ is a Lebesgue point of U_α for all α and $\underline{z} \in V_\alpha$. Recall that conjecturally, $\mathcal{H}^{k(\alpha)}(V_\alpha) > 0$, for all α; see Conjecture 4.63. Under the assumptions of Theorem 14.1, we have $\dim(V_\alpha) \leq k(\alpha)$.

Theorem 14.2. *Let (Z, μ) satisfy* (0.1), *and* (4.3), *for some $1 < p < \infty$, with Z complete. Assume in addition, that Z admits an isometric imbedding $i : Z \to \mathbf{R}^N$, for some $N < \infty$. If $\mathcal{H}^{k(\alpha)}(V_\alpha) > 0$, then V_α is $k(\alpha)$-rectifiable. In particular, if* (13.9) *holds with $k = k(\alpha)$, for all $\underline{z} \in V_\alpha$, then V_α is $k(\alpha)$-rectifiable.*

Proof. Since by Theorem 14.2, for all $\underline{z} \in V_\alpha$, the set $i(Z)$ has a unique approximate tangent plane of dimension, $k(\alpha)$, this follows from a standard characterization of rectifiable sets; see [F]. □

As an immediate consequence of Theorem 14.1, we get the following bi-Lipschitz nonimbedding theorem.

Theorem 14.3. *Let (Z,μ) satisfy (0.1), and (4.3), for some $1<p<\infty$, with Z complete. Assume in addition, that Z admits a bi-Lipschitz imbedding in $\mathbf{R}^N$, for some $N<\infty$. Then for all sets, U_α, as in Theorem 4.38, there exists a subset, V_α, with $\mu(U_\alpha\setminus V_\alpha)=0$, such that for all $\underline{z}\in V_\alpha$, every tangent cone, $Z_{\underline{z}}$, is bi-Lipschitz to $\mathbf{R}^{k(\alpha)}$.*

It is not difficult to check that Carnot-Caratheodory spaces, Laakso spaces and the boundaries of 2-dimensional hyperbolic buildings all violate the necessary condition of Theorem 14.3. On the other hand, in [HaHe], an Ahlfors n-regular space satisfying a Poincaré inequality of type (1,1) is constructed for all positive integers n. These spaces are rectifiable and are believed to admit bi-Lipschitz imbeddings in $\mathbf{R}^{n+1}$.

15 (ϵ,δ)-Inequalities and Thickly Minimally Connected Spaces

Let Z be a rectifiably connected metric space and let μ be a Borel regular measure on Z. In this section, we formulate certain inequalities which we call (ϵ,δ)-*inequalities*. Under the assumption that Z satisfies an (ϵ,δ)-inequality for all $\epsilon,\delta>0$, with constant, $C_{\epsilon,\delta}$, which might *blow up* as $\epsilon,\delta\to 0$, it is essentially trivial to obtain a *quantitative* version of the equality, $g_f=\operatorname{Lip} f$; see Lemma 15.6. This plays a key role in section 16.

We also define the concept of being (ϵ,δ)-*thickly connected*. It follows easily that a space which is (ϵ,δ)-thickly connected admits an (ϵ,δ)-inequality; see Theorem 15.17 and compare also [Se3], where thick families of paths are used to obtain Poincaré inequalities and sufficient conditions are given for the existence of such thick families. We call a space *thickly minimally connected* if it is (ϵ,δ)-thickly connected for all $\epsilon,\delta>0$.

The spaces constructed by T.J. Laakso in [L], provide the first family of examples of length spaces of arbitrary Hausdorff dimension, $1\le d<\infty$, which carry a measure satisfying (0.1) and which are thickly minimally connected. An inequality which we call the *segment inequality* is also satisfied; see (15.21). The segment inequality, a strong version of the Poincaré inequality, implies the existence of a strong version of (ϵ,δ)-inequality, for all $\epsilon,\delta>0$. Laakso's spaces are Ahlfors-regular and the relevant measure is Hausdorff measure.

We begin by noting that the definition of the function F_g of section 5 can be generalized as follows. Given a nonnegative Borel function, $g: Z\to[0,\infty]$, we define $F_{g,\epsilon} : Z \times Z \to [0,\infty]$, by

$$F_{g,\epsilon}(z_1, z_2) = \inf_c \int_0^\ell g(c(s))\, ds\,, \tag{15.1}$$

where the infimum is over all curves, c, from z_1 to z_2, of length at most $(1+\epsilon)\overline{z_1, z_2}$.

REMARK 15.2. We have $F_{g,\infty} = F_g$. However, for $\epsilon < \infty$, the function, $F_{g,\epsilon}$ does not define a pseudometric in general, since the triangle inequality need not hold. As a consequence, even if the function, g, is bounded, it need not be an upper gradient for $F_{g,\epsilon}$. On the other hand, if g is an upper gradient for f, then $|f(z_1) - f(z_2)| \leq F_{g,\epsilon}(z_1, z_2)$; compare section 5.

Let $\rho : Z \times Z \to [0,\infty)$ be given by $\rho(z_1, z_2) = \overline{z_1, z_2}$ and let c denote a constant function. The function, $F_{g,\epsilon}$, satisfies

$$F_{c+h,\epsilon} \leq c(1+\epsilon)\rho + F_{h,\epsilon}\,; \tag{15.3}$$

compare (5.11).

In the following Definition 15.4, we put $\overline{\underline{z}_1, \underline{z}_2} = r$.

DEFINITION 15.4. The metric space, Z, satisfies an (ϵ,δ)-inequality if there exists $C_{\epsilon,\delta} < \infty$, such that for all g and $\underline{z}_1, \underline{z}_2 \in Z$, there exist $\underline{z}_1', \underline{z}_2'$, with $\overline{\underline{z}_j, \underline{z}_j'} \leq \delta r$, $j = 1, 2$, and

$$F_{g,\epsilon}(\underline{z}_1', \underline{z}_2') \leq C_{\epsilon,\delta}\, r \fint_{B_{(1+\epsilon)(1+2\delta)r}(\underline{z}_1)} g\, d\mu\,. \tag{15.5}$$

Our basic observation is that if an (ϵ,δ)-inequality holds, then the term, $F_{h,\epsilon}$, in (15.3), can be estimated by applying (15.5) with g replaced by h. This is of particular interest in situations in which the normalized L_1 norm of h is small, e.g. if $\underline{z}$ is a Lebesgue point of the function g and the distance, r, from $\underline{z}_1$, to $\underline{z}_2$ is sufficiently small. In this circumstance, h can be viewed as the "lower order term" in (15.3). Note that for the "leading term", the inequality, $F_{c,\epsilon} \leq c(1+\epsilon)\rho$, becomes sharp as $\epsilon \to 0$. Thus, any possible lack of sharpness in the constant in the (ϵ,δ)-inequality, effects only the lower order term.

LEMMA 15.6. *Let (Z,μ) satisfy an (ϵ,δ)-inequality, with $\epsilon, \delta \leq 1$, and let $f : Z \to \mathbf{R}$ be a Lipschitz function, with*

$$\mathbf{Lip}\, f \leq L\,. \tag{15.7}$$

If g is a generalized upper gradient for f, such that for some Borel function, $h \geq 0$, and constant, $c \leq L$,

$$g(z) \leq c + h \qquad (\text{on } B_{(1+\epsilon)(1+2\delta)r}(\underline{z}))\,, \tag{15.8}$$

then for $\overline{\underline{z}, z} = r$,

$$\frac{|f(z) - f(\underline{z})|}{r} \leq c + 4(\epsilon + \delta)L + C_{\epsilon,\delta} \fint_{B_{(1+\epsilon)(1+2\delta)r}(\underline{z})} h \, d\mu \, . \tag{15.9}$$

Proof. Assume first that f is Lipschitz and g is a generalized upper gradient for f.

For $\underline{z}', z'$ as in Definition 15.4, we have by (15.3),

$$\begin{aligned} \big|f(z) - f(\underline{z})\big| &\leq 2\delta Lr + \big|f(z') - f(\underline{z}')\big| \\ &\leq 2\delta Lr + F_{g+\epsilon}(\underline{z}', z') \\ &\leq 2\delta Lr + c(1+\epsilon)(1+2\delta)r + F_{h,\epsilon}(\underline{z}', z') \\ &\leq (c + 4\delta L + 4\epsilon L)r + F_{h,\epsilon}(\underline{z}', z') \, . \end{aligned} \tag{15.10}$$

By using (15.5) to estimate the term, $F_{h,\epsilon}(\underline{z}', z')$, we get (15.9).

If more generally, g is an generalized upper gradient of f, we take $f_i \xrightarrow{L_p} f$, $g_i \xrightarrow{L_p} g$, with g_i an upper gradient for f_i and apply the previous argument. In this way, the proof is easily completed. □

Corollary 15.11. *If (Z, μ) satisfies an (ϵ, δ)-inequality for all $\epsilon, \delta > 0$, then $g_f = \operatorname{Lip} f$, for all Lipschitz functions f. In particular, if, under the assumptions of Lemma 15.6,*

$$C_{\epsilon,\delta} = C^{N+1}(\epsilon^{-N} + \delta^{-N}) \, , \tag{15.12}$$

then

$$\frac{|f(z) - f(\underline{z})|}{r} \leq c + 8CL^{\frac{N}{N+1}} \left(\fint_{B_{(1+\epsilon)(1+2\delta)r}(\underline{z})} h \, d\mu \right)^{1/(N+1)} . \tag{15.13}$$

Proof. This follows by using (15.12) and equating the second and third terms on the right-hand side of (15.9). □

In the next corollary, we consider (Z, μ) satisfying (0.1), (4.3). We let ρ_0 denote the length space metric canonically associated to the underlying metric and let metric, $\overline{\rho}$ be as in Remark 12.6.

Corollary 15.14. *If (Z, μ) satisfies (0.1), and (4.3), for some $1 < p < \infty$ and for the metric, ρ_0, an (ϵ, δ)-inequality holds for all $\epsilon, \delta > 0$, then $\rho_0 = \overline{\rho}$.*

Proof. We have $\operatorname{Lip}_{\rho_0} \overline{\rho} \leq \operatorname{Lip}_{\rho} \overline{\rho} = 1$, for μ-a.e. z. It follows from Lemma 15.6, that $\operatorname{Lip}_{\rho_0} \overline{\rho}(\underline{z}, z) \leq 1$, for *all* z. Clearly, this suffices to complete the proof. □

Definition 15.15. A pair, (Z, μ), with Z a length space, is called (ϵ, δ)-*thickly connected*, if for all $\underline{z}_1, \underline{z}_2 \in Z$, there exists,

- a measure space, $X = X_{\epsilon,\delta}(\underline{z}_1, \underline{z}_2)$, $\theta = \theta_{\epsilon,\delta}(\underline{z}_1, \underline{z}_2)$ with $r \cdot \theta(X) = \mu(B_{(1+\epsilon)(1+2\delta)r}(\underline{z}_1))$,
- a subset $U = \{(s,x) \,|\, 0 \leq s \leq \ell(x)\} \subset [0,\infty)$, for some measurable function, $\ell : X \to [0,\infty)$,
- a measurable map, $\phi : U \to Z$, such that $\phi \,|\, [0,\ell(x)]$ is a curve parameterized by arclength, for all $x \in X$,

such that for all $x \in X$,

i) $\phi((0,x)) \in B_{\delta r}(\underline{z}_1)$, $\phi((\ell(x),x)) \in B_{\delta r}(\underline{z}_2)$, where $r = \overline{\underline{z}_1, \underline{z}_2}$,
ii) $\ell(x) \leq (1+\epsilon)\overline{\phi((0,x)), \phi((\ell(x),x))}$,
iii) for some $0 < C'_{\epsilon,\delta} < \infty$ and all $\underline{z}_1, \underline{z}_2$,

$$\phi_*(ds \times \theta) \leq C'_{\epsilon,\delta} \cdot \mu\,. \tag{15.16}$$

It is clear from the definition that say a $(2, 1/4)$-minimally connected length space satisfies (0.1) for some constant $\kappa = \kappa(C_{2,1/4})$. It is also clear that any *ball* whose complement contains a point at a definite distance from the boundary, satisfies an isoperimetric inequality in the sense of [BHo], with a definite constant depending on $C_{2,1/4}$.

Theorem 15.17. *If (Z,μ) is (ϵ,δ)-thickly connected then an (ϵ,δ)-inequality holds, with constant, $C_{\epsilon,\delta} = C'_{\epsilon,\delta}$.*

Proof. By i)–iii) above, we get

$$\int_X \left(\int_0^{\ell(x)} g \circ \phi(x,s) ds \right) d\theta \leq C'_{\epsilon,\delta} \int_{B_{(1+\epsilon)(1+2\delta)r}(\underline{z}_1)} g \, d\mu\,. \tag{15.18}$$

From this, together with the properties preceding i)–iii), the conclusion follows. □

Definition 15.19. A pair, (Z,μ), is called *thickly minimally connected* if it is (ϵ,δ)-thickly connected, for all $\epsilon, \delta > 0$.

Note that a (complete) thickly minimally connected space is in particular, a length space and satisfies the conclusion of Corollary 15.11.

Remark 15.20. It is not clear that thickly minimally connected spaces satisfy a Poincaré inequality. However, for the Poincaré inequality to hold, it suffices to assume the condition on the existence of thick families of paths given in [Se3]. Thus, it might be natural to define a notion of *thickly connected* space, whose definition requires this condition and thick minimal connectedness. In view of Lemma 15.6 and Theorem 15.17, for a thickly minimally connected space, the existence of a Poincaré inequality is equivalent to the existence of an isoperimetric inequality for *arbitrary* sets; see [BHo].

It follows immediately from directionally restricted relative volume comparison that for riemannian manifolds, M^n, diameter at most d and Ricci curvature, $\mathrm{Ric}_{M^n} \geq -(n-1)$, is thickly minimally connected and (15.12), holds with $C = C(n,d)$, $N = 2n$.

For manifolds with $\mathrm{Ric}_{M^n} \geq -(n-1)$ and their limit spaces, the existence of (ϵ,δ)-inequalities, for all $\epsilon,\delta > 0$, can also be deduced from the doubling condition and the *segment inequality* which is a strong version of the Poincaré inequality; compare [ChCo1,3].

Fix $1 \leq p < \infty$. We say that the segment inequality holds if there exists $\underline{\tau} = \underline{\tau}(d)$ such that for all $\underline{z}$ and $0 < r \leq d$,

$$\fint_{B_r(\underline{z})\times B_r(\underline{z})} F_{g,0}\, d(\mu\times\mu) \leq \underline{\tau} r \fint_{B_{2r}(\underline{z})} g\, d\mu\,. \tag{15.21}$$

16 Quantitative Behavior of Almost Generalized Linear Functions

In this section, we assume that Z is a complete length space satisfying an ϵ,δ-inequality, for all (ϵ,δ). We will give quantitative versions of some of our previous estimates. The main results are Theorem 16.32. and Proposition 16.43 (the latter of which requires that (4.3) holds, for some $1 \leq p < \infty$). These have applications in the context of [ChCoMi] and [ChCo3].

We begin by showing that functions which almost satisfy the assumptions of Theorems 8.5, 8.6, almost satisfy the conclusions of those theorems. Then we give some observations which yield an explicit estimate on the function, r_f, on which the function, Ψ, of (16.42) depends.

Remark 16.1. Our discussion could equally well be carried out under the assumption that (0.1) and the segment inequality, (15.21), hold.

Our next two results, Lemmas 16.2, 16.11, are simple elaborations of Lemma 15.6. Recall that the existence of a $(2,1/4)$-inequality implies that (0.1) holds, with $\kappa = \kappa(C_{2,1/4})$.

Lemma 16.2. *Let (Z,μ) satisfy an (ϵ,δ)-inequality, for all $\epsilon,\delta > 0$. Let $f : Z \to \mathbf{R}$, be Lipschitz, with*

$$\mathbf{Lip}\, f \leq L\,. \tag{16.3}$$

If g is a generalized upper gradient for f, such that for some $\underline{z} \in Z$, $r > 0$, $h \geq 0$, $c \leq L$,

$$g(z) \leq c + h \qquad (on\, B_{5r}(\underline{z})) \tag{16.4}$$

and

$$\fint_{B_{5r}(\underline{z})} h\,d\mu < 2^{-3\kappa}\lambda c\,. \tag{16.5}$$

Then for all $\epsilon > 0$ *and* $\theta \le 1$, *there exists* $U \subset B_r(\underline{z})$, *such that*

$$\frac{\mu(U)}{\mu(B_{5r}(\underline{z}))} \le \theta, \tag{16.6}$$

and for $\underline{z}_1, \underline{z}_2 \in B_r(\underline{z}) \setminus U$,

$$\frac{|f(\underline{z}_1) - f(\underline{z}_2)|}{\overline{\underline{z}_1, \underline{z}_2}} \le (1+\chi)c\,, \tag{16.7}$$

where

$$\chi = 8\epsilon\tfrac{L}{c} + C_{\epsilon,\epsilon}\tfrac{\lambda}{\theta}\,. \tag{16.8}$$

Proof. Without loss of generality, we can assume say $\epsilon < 1/4$, $\delta < 1/2$.

Fix $\theta < 1$. By the standard covering argument based on (0.1), there exists, $\{B_{6s_i}(z_i)\}$, such that if we put $U = \cup_i B_{6s_i}(z_i) \cap B_{5r}(\underline{z})$, then (16.6) holds and for $z \in B_{5r}(\underline{z}) \setminus U$, and $B_s(z) \subset B_{5r}(\underline{z})$, we have

$$h_{z,s} \le 2^{3\kappa}\tfrac{c}{\theta}\lambda\,. \tag{16.9}$$

By Lemma 15.6, this suffices to complete the proof. □

Remark 16.10. Under the assumption that (4.3) holds, there is a version of Lemma 16.2, in which the bound, (16.3), on $\mathbf{Lip}\, f$ is omitted.

Lemma 16.11. *Let* (Z,μ) *satisfy an* (ϵ,δ)*-inequality, for all* $\epsilon, \delta > 0$. *Assume* (16.3), (16.4) *and*

$$\fint_{B_{5r}(\underline{z})} h\,d\mu < \frac{2^{-3\kappa}}{C_{\psi_1 L/160c,\psi_1 L/160c}}\left(\frac{\psi_1}{48}\right)^{\kappa+1} c\,. \tag{16.12}$$

Then there exists $\breve{f} : B_r(\underline{z}) \to \mathbf{R}$, *with*

$$\mathbf{Lip}\, \breve{f} \le c\,, \tag{16.13}$$

and

$$|f - \breve{f}|_{L_\infty} \le \psi_1 L r\,. \tag{16.14}$$

Proof. If in Lemma 16.2, we choose λ such that two terms on the right-hand side of (16.8) are equal, we get $\chi = 16\epsilon\frac{L}{c} = \frac{1}{10}\psi_1$, where by definition, $\psi_1 = 160\epsilon\frac{L}{c}$. If we also choose θ such that $5\chi = 24\cdot\theta^{1/\kappa}$, then the right-hand side of (16.5) is easily seen to be greater than the right-hand side of (16.12).

Let U, be as in Lemma 16.2. By MacShane's lemma (see (8.2), (8.3)) we can extend $f\,|\,U$ to a function, $f_* : B_{5r}(\underline{z}) \to \mathbf{R}$, without increasing its Lipschitz constant. Put

$$\breve{f} = \frac{1}{1+\chi} f_* + \frac{\chi}{1+\chi} f_*(\underline{z})\,. \tag{16.15}$$

Clearly, (16.13) holds. Moreover, since $\breve{f}(\underline{z}) = f_*(\underline{z})$ and

$$\mathbf{Lip}\,(\breve{f} - f_*) \le \chi c\,, \tag{16.16}$$

we get

$$|\breve{f} - f_*|_{L_\infty} \le 5\chi c r\,. \tag{16.17}$$

By (16.3), (16.13), we have $\mathrm{Lip}\,(f - f_*) \le 2L$. From (0.1), (16.6), it follows that the set on which $f - f_*$ vanishes is $12 \cdot \theta^{1/\kappa}$-dense. Thus, with (16.17), we get

$$\begin{aligned} |f - \breve{f}|_{L_\infty} &\le (5\chi + 24\theta^{1/\kappa})Lr \\ &= \psi_1 L r\,. \end{aligned} \tag{16.18}$$

This completes the proof. □

In Lemma 16.19 and Theorem 16.32 below, the functions k^*, k_*, and $\breve{f}^*, \breve{f}_*$, are defined as in (8.2), (8.3), where $A = \partial B_r(\underline{z})$.

Lemma 16.19. *Let* (Z,μ) *satisfy* (0.1) *and let* $\partial B_r(\underline{z}) \not\equiv \emptyset$. *Let* $k : \partial B_r(\underline{z}) \to \mathbf{R}$, *satisfy*

$$\mathbf{Lip}\, k \le c \qquad (c \le L)\,, \tag{16.20}$$

Assume there exists $v : \partial B_r(\underline{z}) \to \mathbf{R}$, *such that*

$$\mathbf{Lip}\, v \le L\,, \tag{16.21}$$

$$\sup_{\partial B_r(\underline{z})} |v - k| \le \psi_1 L r\,, \tag{16.22}$$

and for any Lipschitz function, $\tilde{v} : B_r(\underline{z}) \to \mathbf{R}$, *with* $\tilde{v} \,|\, \partial B_r(\underline{z}) = v$,

$$c - \psi_2 L \le \left(⨍_{B_r(\underline{z})} (\mathrm{Lip}\,\tilde{v})^p \, d\mu \right)^{1/p}. \tag{16.23}$$

Then

$$k^* \le k_* + 4\big(\psi_2 + 5(2\psi_1)^{\beta/p}\big)^{1/p\kappa} L r\,, \tag{16.24}$$

with β *is as in* (6.14).

Proof. Let $v', \tilde{k}$ be extensions of v, k to $\overline{B_r(\underline{z})}$, with $\mathrm{Lip}\, v' \le L$ and $\mathrm{Lip}\, \tilde{k} \le c$.

Fix $\eta > 0$ and let $\phi : \overline{B_r(\underline{z})} \to [0,1]$ be the Lipschitz function such that $\phi \,|\, \partial \overline{B_r(0)} \equiv 0$, $\phi \,|\, B_{(1-\eta)r}(\underline{z}) \equiv 1$ and $|\mathrm{Lip}\,\phi|_{L_\infty} \le (\eta r)^{-1}$. By taking $\tilde{v} = \tilde{k} + (1-\phi)(v' - \tilde{k})$ in (16.23), and employing Lemma 1.7, (6.13), (16.21), (16.22), we get

$$\begin{aligned} \left(⨍_{B_r(\underline{z})} (\mathrm{Lip}\,\tilde{v})^p \, d\mu \right)^{1/p} &\le \left(⨍_{B_r(\underline{z})} (\mathrm{Lip}\,\tilde{k})^p \, d\mu \right)^{1/p} \\ &\quad + (2\eta)^{\beta/p}(\psi_1 L \eta^{-1} + 4L)\,. \end{aligned} \tag{16.25}$$

Taking $\eta = \psi_1$, and using (16.23), gives

$$c - \psi_2 L - 5(2\psi_1)^{\frac{\beta}{p}} L \le \left(\fint_{B_r(\underline{z})} (\mathrm{Lip}\, \tilde{k})^p \, d\mu \right)^{1/p} . \tag{16.26}$$

Since $\mathrm{Lip}\, \tilde{k} \le c$, we have

$$(\mathrm{Lip}\, \tilde{k})^p + (c - \mathrm{Lip}\, \tilde{k})^p \le c^p . \tag{16.27}$$

Raising both sides of (16.26) to the power, p, and using (16.27), easily yields

$$\left(\fint_{B_r(\underline{z})} \left| c - (\mathrm{Lip}\, \tilde{k}) \right|^p d\mu \right)^{1/p} \le 2\bigl(\psi_2 + 5(2\psi_1)^{\beta/p}\bigr)^{1/p} L . \tag{16.28}$$

In particular, this holds for $\tilde{k} = k^*, k_*$.

Assume that for some $w \in B_r(\underline{z})$, we have $k_*(w) + 2\theta r \le k^*(w)$. Put $K = \frac{1}{2}(k_*(w) + k^*(w))$.

Since $\mathrm{Lip}\, k_* \le c$, $\mathrm{Lip}\, k^* \le c$, it follows that the set of points, z, such that

$$k_*(z) \le K \le k^*(z) , \tag{16.29}$$

contains the ball, $B_{\theta r/c}(w)$.

Let the function, $\hat{k}$, given by

$$\hat{k}(z) = \begin{cases} k^*(z) & \text{if } k^*(z) \le K , \\ K & \text{if } k_*(z) \le K \le k_*(z) , \\ k_*(z) & \text{if } K \le k_*(z) . \end{cases} \tag{16.30}$$

By construction, $\mathrm{Lip}\, \hat{k} \le c$ and $\hat{k} \,|\, \partial B_r(\underline{z}) = k \,|\, \partial B_r(\underline{z})$. Thus, (16.28) holds, with $\tilde{k}$ replaced by $\hat{k}$. Also, since $\mathrm{Lip}\, \hat{k} \,|\, B_{\theta r/c}(w) \equiv 0$, from (0.1) and (16.28) we get

$$c = \left(\fint_{B_{\theta r/c}(w)} |\mathrm{Lip}\, \hat{k} - c|^p \, d\mu \right)^{1/p} \le \left(\frac{2c}{\theta} \right)^{\kappa} 2\bigl(\psi_2 + 5(2\psi_1)^{\beta/p}\bigr)^{1/p} L . \tag{16.31}$$

This gives (16.24). □

By combining Lemma 16.19 with Lemma 16.11, taking $v = f \,|\, \partial B_r(z)$ and $k = \breve{f} \,|\, \partial B_r(z)$, we immediately obtain:

Theorem 16.32. *Let (Z, μ) satisfy (0.1) and an (ϵ, δ)-inequality, for all $\epsilon, \delta > 0$. Let $f : B_{5r}(\underline{z}) \to \mathbf{R}$ satisfy (16.3), (16.4), (16.12). Assume in addition that for any Lipschitz function, $\tilde{f} : B_r(\underline{z}) \to \mathbf{R}$, with $\tilde{f} \,|\, \partial B_r(\underline{z}) = f$,*

$$c - \psi_2 L \le \left(\fint_{B_r(\underline{z})} (\mathrm{Lip}\, \tilde{f})^p \, d\mu \right)^{1/p} . \tag{16.33}$$

Then the function, $\breve{f}$, *satisfying* (16.13), (16.14), *whose existence is guaranteed by Theorem 16.11, also satisfies*

$$\breve{f}^* \le \breve{f}_* + 4\big(\psi_2 + 5(2\psi_1)^{\beta/p}\big)^{1/p\kappa} Lr \qquad (\text{on } B_r(\underline{z}))\,, \quad (16.34)$$

with β *is as in* (6.14). *In particular, the conclusions of Lemma 8.17 hold, with* f *replaced by* $\breve{f}$.

Remark 16.35. By arguing as in section 8 (see in particular, Theorem 8.11) we can employ Theorem 16.32 to show directly that on $\mathbf{R}^n$, asymptotically generalized linear functions converge in the limit as $r \to 0$, to linear functions.

Remark 16.36. In view of Remark 16.35, the most direct route to obtaining the classical theorem of Rademacher via the results of this paper, would be the following. Use Theorems 3.7, 15.17 and Corollary 15.11 to show that on $\mathbf{R}^n$, a Lipschitz function, f, is asymptotically generalized linear and in addition, $g_f(\underline{x}) = \mathrm{Lip}\, f(\underline{x})$, for a.e. $\underline{x}$. Then use Theorem 16.32 and the argument of Theorem 8.11 to show that f is asymptotically linear (as defined after (0.4)). Finally, use Lemmas 4.32, 4.35 to show the uniqueness of limit functions, $f_{0,\underline{x}}$, as in (0.2)–(0.4).

Now, we give a quantitative version of Lemma 6.30, in which the function, r_f, of that lemma is replaced by a function, R_f, depending only on the parameters, η, δ. Our next lemma, relies on Lemma 15.6 and is otherwise completely analogous to Lemma 6.30. Hence the proof is omitted. The function, Ψ depends only on the specified parameters, κ, τ, L, ξ; compare section 10.

Let $f : B_R(\bar{z}) \to \mathbf{R}$ be Lipschitz. It follows from Lemma 6.30 that for all $1 > \eta, \psi > 0$, there exists $0 < R_f(\eta, \delta)$, and $Z_f(\eta, \psi) \subset B_R(\bar{z})$, such that

$$\mu\big(Z_f(\eta,\psi)\big) > (1-\eta)\mu\big(B_R(\bar{z})\big)\,, \quad (16.37)$$

and for all $\underline{z} \in Z_f(\eta, \psi)$, we have

$$\fint_{B_{3r}(\underline{z})} \big|\mathrm{Lip}\, f(z) - \mathrm{Lip}\, f(\underline{z})\big|^p \, d\mu < \psi \qquad (r \le R_f(\eta,\psi))\,. \quad (16.38)$$

Lemma 16.39. *Let* (Z, μ) *satisfy* (0.1) *and an* (ϵ, δ)*-inequality, for all* $\epsilon, \delta > 0$. *Let* $f : B_R(\bar{z}) \to \mathbf{R}$ *be a Lipschitz function satisfying*

$$\mathbf{Lip}\, f \le L\,. \quad (16.40)$$

Then for all $1/2 > \eta, \xi > 0$, *there exists a finite collection of balls,* $\{B_{r_j}(z_j)\}$, *such that* (6.31), (6.32), (6.34) *hold. In addition,*

$$r_j > C\big(\kappa, R_f(\eta,\psi)\big)\,, \quad (16.41)$$

and for all $\underline{z}_{1,j}, \underline{z}_{2,j} \in B_{r_j}(z_j)$*, such that* $\overline{\underline{z}_{1,j}, \underline{z}_{2,j}} > \xi r_j$*, we have*

$$\frac{|f(\underline{z}_{1,j}) - f(\underline{z}_{2,j})|}{\overline{\underline{z}_{1,j}, \underline{z}_{2,j}}} < \mathrm{Lip}\ f(z_j) + \Psi(\delta|\kappa, C_{\epsilon,\delta}, L, \xi)\,. \tag{16.42}$$

Assume that for some $q > 0$, the function, $\mathrm{Lip}\, f$, lies in $H_{1,q}$. By applying Lemma 16.39, to the function, $\mathrm{Lip}\, f$, we immediately obtain the following estimate for the function, R_f.

PROPOSITION 16.43. *Let* (Z, μ) *satisfy* (0.1)*, and* (4.3)*, for some* $1<p<\infty$*. If* $f : Z \to \mathbf{R}$ *is a Lipschitz function satisfying (16.40) and for some* $q > 0$*,*

$$|\mathrm{Lip}\, f|_{H_{1,q}} \leq A\,, \tag{16.44}$$

then

$$R_f(\eta, \psi) \geq C_q(\kappa, \tau, L, A, \eta, \psi)\,. \tag{16.45}$$

17 Appendix: Quasi-convexity

We are indebted to Stephen Semmes for explaining to us the proof of Theorem 17.1 below. The proof which follows is an exposition (employing (4.22)) of Semmes' argument; compare also [DSe2] and for Semmes' exposition, see Lemma 2.38 of [Se5].

Theorem 17.1. *Let* (Z, μ) *satisfy* (0.1)*, and* (4.3)*, for some* $1 < p < \infty$*, with* Z *complete. Then* Z *is* $c(\kappa, C)$*-quasi-convex.*

Proof. Let Z be a metric space. We say the $z, w \in Z$ lie in the same ϵ-component of Z if there exists a *finite* sequence $z_0, \ldots, z_N$, with $z_0 = z$, $z_N = w$ such that $\overline{z_i, z_{i+1}} \leq \epsilon$, for all $1 \leq i \leq N-1$. Clearly, the ϵ-component of $\underline{z}$ is a closed set containing $\overline{B_\epsilon(\underline{z})}$. Moreover, the relation of lying in the same ϵ-component is an equivalence relation.

If z, w lie in different ϵ-components, then it is obvious that there exists no rectifiable curve from z, w. Thus, the function, $g \equiv 0$, is an upper gradient for the characteristic function of any component.

The assumption that Z carries a nontrivial measure, μ, satisfying (0.1), implies every ϵ-component has positive measure. By applying (4.3) to the characteristic function of some component, it follows that there is just one ϵ-component in this case.

For $\epsilon > 0$, define the *regularized distance function* from $\underline{z}$ by

$$\rho_{\underline{z},\epsilon}(z) = \inf_{\{z_i\}} \sum_{i=0}^{N-1} \overline{z_i, z_{i+1}}\,, \tag{17.2}$$

where the inf is taken over all finite sequences with $z_0 = \underline{z}$, $z_N = z$ such that $\overline{z_i, z_{i+1}} \leq \epsilon$, for all $0 \leq i \leq N-1$. For $\overline{z_1, z_2} < \epsilon$, we have $|\rho_{\underline{z},\epsilon(z)}(z_1) - \rho_{\underline{z},\epsilon(z)}(z_2)| \leq \overline{z_1, z_2}$. In particular, $\mathrm{Lip}\, \rho_{\underline{z},\epsilon}(z) \leq 1$, for all z. Note also that, for fixed $\underline{z}, z$, the function, $\rho_{\underline{z},\epsilon}(z)$, is a decreasing function of ϵ.

Clearly, for all $\epsilon > 0$, the function, $g \equiv 1$, is an upper gradient for $\rho_{\underline{z},\epsilon}$. Thus, by (4.22), for all $0 < r < \infty$, we have

$$\int_{B_r(\underline{z})} \rho_{\underline{z},\epsilon}\, d\mu \leq 2^{\kappa+1} C r \mu(B_r(\underline{z}))\,. \tag{17.3}$$

By the dominated convergence theorem, there exists $\rho_{\underline{z},0}$ such that on $B_r(\underline{z})$, we have $\rho_\epsilon \xrightarrow{L_1} \rho_{\underline{z},0}$, as $\epsilon \to 0$.

Since Z is complete, the existence of the nontrivial measure, μ, satisfying (0.1), implies that that closed balls of finite radius are compact. Thus, a standard limiting argument implies that if $\rho_{\underline{z},0}(z) < \infty$, then there exists a rectifiable curve, c, from $\underline{z}$ to z of length $\rho_{\underline{z},0}(z)$. It is also clear that there exists no shorter rectifiable curve from $\underline{z}$ to z. Thus, in the terminology of section 5, we have $\rho_{\underline{z},0}(z) = F_1(\underline{z}, z)$.

It follows from (17.3) together with (0.1), that for all z, there exists $\underline{z}_1$, with $\overline{\underline{z}_1, z} \leq \frac{1}{2}\overline{\underline{z}, z}$, such that there exists a curve, c_1, from $\underline{z}$ to $\underline{z}_1$, of length at most $\frac{1}{2}c(\kappa, C)\overline{\underline{z}, z}$. By repeating this construction, with $\underline{z}_1$ in place of $\underline{z}$, and proceeding by induction, we obtain a sequence of points, $\underline{z}_i$, and curves, c_i, from $\underline{z}_{i-1}$ to $\underline{z}_i$, such that the infinite union, $c_1 \cup c_2 \cdots$, is a rectifiable curve from $\underline{z}$ to z, of length at most $c(\kappa, C)\overline{\underline{z}, z}$. $\square$

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Jeff Cheeger
Courant Institute of Mathematical Sciences
251 Mercer Street
New York, NY 10012
cheeger@cims.nyu.edu

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