MAT-544 Schul Midterm October 23

First Name: ____________________

Last Name: ____________________

Stony Brook ID: ____________________

Signature: ____________________

Choose **4 out of 5 problems**. (Clearly mark which one you are NOT DOING.) Write coherent mathematical statements and show your work on all problems. If you use a theorem from the book, you must fully state it. If you give an example/construction then you must prove it is such. Please write clearly.

Rules.

1. Start when told to; stop when told to.
2. No notes, books, etc,...
3. Turn off all unauthorized electronic devices (for example, your cell phone).

1	2	3	4	5

TOTAL:

FOR ALL QUESTIONS: Unless stated otherwise, always assume:

- $(\mathbb{X}, \mathcal{M}, \mu)$ is a measure space. $\bar{\mathbb{R}}$ is the extended real number system $\mathbb{R} \cup \{-\infty, +\infty\}$.
- when a function $q : \mathbb{X} \to \bar{\mathbb{R}}$ is said to be measurable, we mean that it is measurable with respect to the σ-algebras $\mathcal{M}$ on $\mathbb{X}$ and $\mathcal{B}$ (the Borel σ-algebra) on $\bar{\mathbb{R}}$; in other words, q is $(\mathcal{M}, \mathcal{B})$-measurable.

1. (10 points) Suppose $f_n : \mathbb{X} \to \bar{\mathbb{R}}$ is a sequence of measurable functions. Let

$$g = \inf f_n \,.$$

and

$$h = \liminf f_n \,.$$

Show that g and h are measurable.

2. (10 points)

   (a) State and prove Fatou's Lemma

   (b) Give a specific example of a situation where the inequality in the statement you gave is not an equality.

3. (10 points) Explicitly construct a Cantor set of positive measure: a set $K \subset \mathbb{R}$ which is compact, totally disconnected and has no isolated points, such that K has positive Lebesgue measure.

4. (10 points)

(a) What is a simple function?

(b) Let $f : \mathbb{X} \to [0, \infty)$ be a measurable function. Show by construction that there is a sequence of *simple* functions $\phi_n : \mathbb{X} \to \mathbb{R}$ such that $\phi_n \to f$ everywhere, and that this limit is uniform on the set $\{x : f(x) < 1000\}$.

5. (10 points)

A mesurable function $f : \mathbb{X} \to \mathbb{R}$ is said to be integrable if $\int |f| d\mu < \infty$. For an integrable function f, define $\|f\|_1 = \int |f| d\mu$. Suppose $f_n : \mathbb{X} \to \mathbb{R}$ are integrable functions and satisfy $f_n \to f$ uniformly.

(a) Show that if $\mu(\mathbb{X}) < \infty$ then $\int f_n \to \int f < \infty$.

(b) Show (by example) that if $\mu(\mathbb{X}) = \infty$ then the conclusion above fails, i.e. we may have $\int f_n \nrightarrow \int f$.