MAT-544 Schul | Final | December 14

First Name: ____________________

Last Name: ____________________

Stony Brook ID: ____________________

Signature: ____________________

There are 8 problems. Write coherent mathematical statements and show your work on all problems. If you use a theorem from the book, **you must fully state it**. If you give an example/construction then you must prove it is such. Please write clearly.

Rules.

1. Start when told to; stop when told to.
2. No notes, books, etc,...
3. Turn off all unauthorized electronic devices (for example, your cell phone).

1 (10pts)	2 (10pts)	3 (20pts)	4 (10pts)	5 (10pts)
6 (10pts)	7 (10pts)	8 (10pts)		TOTAL

FOR ALL QUESTIONS: Unless stated otherwise, always assume:

- $(\mathbb{X}, \mathcal{M}, \mu)$ is a measure space. $\bar{\mathbb{R}}$ is the extended real number system $\mathbb{R} \cup \{-\infty, +\infty\}$.
- when a function $q : \mathbb{X} \to \bar{\mathbb{R}}$ is said to be measurable, we mean that it is measurable with respect to the σ-algebras $\mathcal{M}$ on $\mathbb{X}$ and $\mathcal{B}$ (the Borel σ-algebra) on $\bar{\mathbb{R}}$; in other words, q is $(\mathcal{M}, \mathcal{B})$-measurable.
- a measurable function f is called integrable if $\int |f| < \infty$.
- for an integrable function f we define $\|f\|_1 := \int |f|$.
- m denotes the Lebesgue measure on $\mathbb{R}^n$.

1. (10 points) Let $\mathcal{D}_0$ be the sigma algebra on $\mathbb{R}$ generated by the collection of intervals

$$\{[a, a+1) : a \in \mathbb{Z}\}.$$

Give a precise description of all functions $f : \mathbb{R} \to \mathbb{R}$ which are $\mathcal{D}_0$-measurable, (in other words, functions which are $(\mathcal{D}_0, \mathcal{B})$-measurable).

2. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a positive Borel-measurable function. Define $\nu(E)$ by

$$\nu(E) := \int_E f d\mu.$$

Show that ν is a Borel measure.

3. (20 points) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of Lebesgue-integrable functions. For each of the statements below, prove it or find a counterexample.

   (a) If $f_n \to f$ almost everywhere, then a subsequence converges to f in L^1.

   (b) If $f_n \to f$ in measure, then a subsequence converges to f almost everywhere.

   (c) If $f_n \to f$ in measure, then $f_n \to f$ in L^1.

   (d) If $f_n \to f$ in L^1, then a subsequence converges to f in measure.

(extra page)

4. (10 points)

(a) State the Hahn-Jordan decomposition theorems.

(b) Show that if ν is a signed measure, and $\nu = \mu - \lambda$, the difference of two (positive) measures, then $\nu^+ \leq \mu$ and $\nu^- \leq \lambda$.

(c) Give an example where the inequalities in part (b) are not equalities.

5. (10 points) Let m^n be the Lebesgue measure on $\mathbb{R}^n$. Let $E \subset \mathbb{R}^2$ be a measurable set satisfying $m^2(E) = 3$. For $x \in \mathbb{R}$, let

$$E_x := \{y : (x, y) \in E\}.$$

We know that for almost every $x \in \mathbb{R}$ the set E_x is measurable.

Show that

$$m^1\{x : E_x \textit{ measurable}, m^1(E_x) > \lambda\} < \frac{3}{\lambda}.$$

6. (10 points) Suppose μ is a finite Borel measure on $\mathbb{R}$, which is singular to Lebesgue measure. Show that except for a set of Lebesgue measure zero, we have

$$\lim_{r \to 0} \frac{\mu([x-r, x+r])}{2r} = 0\,.$$

Note: you may use any result from class or the textbook provided you quote it precisely.

7. (10 points) Recall a FACT: For a Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$, and $x \in \mathbb{R}$, define

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f dm\,.$$

Let E_λ be defined as

$$E_\lambda := \{x \in \mathbb{R} : Mf(x) > \lambda\}.$$

Then there is a constant C, independent of f, such that

$$m(E_\lambda) < \frac{C\|f\|_1}{\lambda}\,.$$

For a mesurable set $E \subset \mathbb{R}$, define

$$A_r(x) = \frac{1}{2r}\mu(B(x,r) \cap E).$$

Let $\mu(E) \in (0,\infty)$. **Use the above fact** to show that for almost every $x \in E$ we have

$$\lim_{r\to 0} A_r(x) = \chi_E(x)\,.$$

8. (10 points) Suppose a set $E \subset \mathbb{R}^3$ satisfies that for every $x \in \mathbb{R}^3$ and $r > 0$ there exists a point $z \in B(x, r)$ such that $E \cap B(z, r) \cap B(x, 2r) = \emptyset$. Show that $m(E) = 0$, where m is the Lebesgue measure on $\mathbb{R}^3$.