

# ORDINARY DIFFERENTIAL EQUATIONS

## NOTES FOR MAT 550

DROR VAROLIN

### 1. DEFINITION OF ODE

DEFINITION 1.1. Let  $D \subset \mathbb{R}^n$  be a domain, i.e., an open connected set.

- (i) A time-dependent vector field on  $D$  is a pair consisting of a domain  $V \subset D \times \mathbb{R}$  together with a continuous map  $F : V \rightarrow \mathbb{R}^n$ .
- (ii) The time-dependent vector field  $F$  is said to be autonomous (or one simply omits the adjective *time-dependent*) if  $V = D \times \mathbb{R}$  and for each  $x \in D$ ,  $F(x, \cdot)$  is constant. That is to say, a vector field on  $D$  is a continuous map  $\xi : D \rightarrow \mathbb{R}^n$ . (In particular,  $\xi(x) = F(x, t)$  for all  $(x, t) \in D \times \mathbb{R}$ .)  $\diamond$

REMARK 1.2. In principal, the above definition can be made for more general measurable vector fields. However, since for a given vector field we will be seeking a measurable function that we will compose with the vector field in question, continuity is a natural assumption.  $\diamond$

Vector fields are the data of ordinary differential equations (ODE). From this data, we want to extract so-called *integral curves*, which we now define.

DEFINITION 1.3. Let  $F$  be a time-dependent vector field on a domain  $V \subset D \times \mathbb{R}$ . An integral curve through  $x \in D$  with initial time  $s$  is an open set  $I_{(x,s)} \subset \mathbb{R}$  containing  $s$ , together with a differentiable curve  $\gamma_{(x,s)} : I_{(x,s)} \rightarrow D$ , such that

- (i)  $\gamma_{(x,s)}(s) = x$ ,
- (ii)  $(\gamma_{(x,s)}(t), t) \in V$  for all  $t \in I_{(x,s)}$ , and
- (iii) one has

$$\frac{d\gamma_{(x,s)}(t)}{dt} = F(\gamma_{(x,s)}(t), t).$$

The central question of ODE is whether, for a given vector field, integral curves exist and, if so, are unique. This question is partially answered in Section 3, after we establish, in Section 2, a simple fact about contraction mappings.

### 2. CONTRACTION MAPPINGS

In the proof of the existence and uniqueness theorem to be stated in the next section, we will need to make use of an iteration scheme due to Picard. The convergence of this iteration scheme depends on the concept of contraction mapping, which we now define.

DEFINITION 2.1. Let  $A \subset X$  be a subset of a metric space. A mapping  $S : A \rightarrow A$  is said to be a *contraction mapping* if there exists some  $r \in (0, 1)$  such that

$$d(Sx, Sy) \leq r \cdot d(x, y)$$

for all  $x, y \in A$ .  $\diamond$

The basic fact about contraction mappings is the following result.

**PROPOSITION 2.2.** *Let  $X$  be a complete metric space and let  $A \subset X$  be a closed subset. Let  $S : A \rightarrow A$  be a contraction mapping. Then  $S$  has a unique fixed point.*

*Proof.* Let  $x \in A$  be any point. Consider the sequence  $\{x_j\}$  defined by

$$x_j := S^{(j)}x, \quad j = 0, 1, 2, \dots$$

where  $S^{(0)} = \text{Id}$  is the identity map and  $S^{(j)} := S \circ S^{(j-1)}$  for all  $j \in \mathbb{N}$ . Then for all  $j < k$  we have

$$d(x_j, x_k) \leq \sum_{\ell=j}^{k-1} d(x_\ell, x_{\ell+1}) \leq \sum_{\ell=j}^{k-1} r^\ell d(x, Sx) = \frac{r^j(1 - r^{k-j-1})}{1 - r} d(x, Sx) \leq \frac{r^j}{1 - r} d(x, Sx).$$

It follows that  $\{x_j\}$  is a Cauchy sequence, and since  $A$  is closed (hence complete), the limit

$$x_* := \lim x_j$$

exists and lies in  $A$ . Since a contraction mapping is continuous,

$$x_* = \lim S^{(j)}x_* = \lim S \circ S^{(j-1)}x_* = S(\lim S^{(j)}x_*) = Sx_*.$$

Thus  $x_*$  is a fixed point of  $S$ .

Finally, if  $y$  is another fixed point of  $S$ , then

$$0 \leq (1 - r)d(x_*, y) = d(Sx_*, Sy) - rd(x_*, y) \leq (r - r)d(x_*, y) = 0.$$

Thus  $y = x_*$ , and the proof is complete. □

### 3. THE EXISTENCE AND UNIQUENESS THEOREM FOR FIRST ORDER ODE

**DEFINITION 3.1.** Let  $f : U \rightarrow \mathbb{R}^n$  be a function defined on a domain  $U \subset \mathbb{R}^m$ . We say that  $f$  is *locally Lipschitz* if for each  $p \in U$  and each  $\varepsilon \in (0, \text{dist}(p, U^c))$  there exists a constant  $K = K_{\varepsilon, p}$  such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all  $x, y \in B(p, \varepsilon) := \{z \in \mathbb{R}^m ; |z - p| < \varepsilon\}$ . ◇

**REMARK 3.2.** Note that any differentiable function is locally Lipschitz, but that the converse is not true, as is shown by the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = |x|$ . ◇

Before stating our next definition and result, we introduce the following notation for the sake of convenience. Let  $D \subset \mathbb{R}^n$  and  $V \subset D \times \mathbb{R}$  be domains. For each  $t \in \mathbb{R}$ , we write

$$V_t = \{x \in D ; (x, t) \in V\}.$$

(Note that this set may be empty.)

**DEFINITION 3.3.** Let  $D \subset \mathbb{R}^n$  and  $V \subset D \times \mathbb{R}$  be domains, let  $F : V \rightarrow \mathbb{R}^n$  be a continuous time-dependent vector field. We say that the function  $F_t : V_t \rightarrow \mathbb{R}^n$  defined by  $F_t(x) := F(x, t)$  is *locally uniformly Lipschitz* on  $V_t$  if  $F_t$  is locally Lipschitz and moreover the Lipschitz constant can be taken locally uniform with respect to  $t$ . ◇

The main theorem of these notes is the following result.

**THEOREM 3.4.** *Let  $D \subset \mathbb{R}^n$  and  $V \subset D \times \mathbb{R}$  be domains, let  $F : V \rightarrow \mathbb{R}^n$  be a continuous time-dependent vector field. Assume that for each  $t \in \mathbb{R}$  the function  $F_t : V_t \rightarrow \mathbb{R}^n$  defined by  $F_t(x) := F(x, t)$  is locally uniformly Lipschitz on  $V_t$ . Then for each  $(x, s) \in V$  there exists an integral curve  $\gamma_{(x,s)} : I_{(x,s)} \rightarrow D$  for  $F$ . Moreover, the set of integral curves possesses the following uniqueness property: if  $\gamma_{(x,s)} : I_{(x,s)} \rightarrow D$  and  $\tilde{\gamma}_{(x,s)} : \tilde{I}_{(x,s)} \rightarrow D$  are two integral curves through  $x$  at time  $s$ , then  $\gamma_{(x,s)}(t) = \tilde{\gamma}_{(x,s)}(t)$  for all  $t \in I_{(x,s)} \cap \tilde{I}_{(x,s)}$ .*

*Proof.* Let  $(x_o, t_o) \in V$  and choose  $\varepsilon > 0$  such that  $F$  is continuous in  $B(x_o, \varepsilon) \times (-\varepsilon, \varepsilon)$  and Lipschitz in the first variable with Lipschitz constant  $K$ , i.e.,

$$|F(x, t) - F(y, t)| \leq K|x - y|$$

for all  $(x, t), (y, t) \in B(x_o, \varepsilon) \times (-\varepsilon, \varepsilon)$ . By continuity there exists a constant  $M > 0$  such that

$$|F(x, t)| \leq M$$

for all  $(x, t) \in B(x_o, \varepsilon) \times (-\varepsilon, \varepsilon)$ .

Choose positive constants  $\alpha$  and  $\beta$  such that

(i) with  $I_\alpha := \{t \in \mathbb{R} ; |t - t_o| \leq \alpha\}$  and  $B_\beta := \{x \in \mathbb{R}^n ; |x - x_o| \leq \beta\}$ ,

$$B_\beta \times I_\alpha \subset B(x_o, \varepsilon) \times (-\varepsilon, \varepsilon),$$

(ii)  $\alpha M < \beta$ , and

(iii)  $\alpha K < 1$ .

Let  $\mathcal{A}$  denote the set of continuous maps  $\phi : I_\alpha \rightarrow \mathbb{R}^n$  such that

$$|\phi(t) - x_o| \leq \beta \quad \text{for all } t \in I_\alpha.$$

Equipping  $\mathcal{A}$  with the uniform norm

$$\|\phi\|_u := \sup_{I_\alpha} |\phi|$$

makes  $\mathcal{A}$  into a closed bounded subset of a Banach (and hence complete metric) space, as we have discussed earlier in the course. Thus  $\mathcal{A}$  is itself a complete metric space with respect to the metric

$$d(\phi, \tilde{\phi}) := \|\phi - \tilde{\phi}\|_u.$$

Consider the operator  $T$  defined by

$$T\phi(t) := x_o + \int_{t_o}^t F(\phi(s), s) ds.$$

Observe first that if  $\phi \in \mathcal{A}$  then clearly  $T\phi$  is continuous and defined on all of  $I_\alpha$ . Moreover, for  $t \in I_\alpha$  one has

$$|T\phi(t) - x_o| \leq M|t - t_o| \leq M\alpha < \beta,$$

and thus  $T\phi \in \mathcal{A}$ . That is to say,

$$T : \mathcal{A} \rightarrow \mathcal{A}.$$

Next, observe that for  $\phi_1, \phi_2 \in \mathcal{A}$  one has

$$\begin{aligned} |T\phi_1(t) - T\phi_2(t)| &= \left| \int_{t_0}^t (F(\phi_1(s), s) - F(\phi_2(s), s)) ds \right| \\ &\leq \int_{t_0}^t K |\phi_1(s) - \phi_2(s)| ds \\ &\leq K\alpha \sup_{I_\alpha} |\phi_1 - \phi_2|. \end{aligned}$$

It follows that for some  $r \in (0, 1)$ ,

$$\|T\phi_1 - T\phi_2\|_u \leq r \|\phi_1 - \phi_2\|_u.$$

Thus  $T : \mathcal{A} \rightarrow \mathcal{A}$  is a contraction mapping, and therefore by Proposition 2.2 it has a unique fixed point  $\phi_* \in \mathcal{A}$ .

Being a fixed point of  $T$ ,  $\phi_*$  satisfies the equation

$$(1) \quad \phi_*(t) = x_o + \int_{t_0}^t F(\phi_*(s), s) ds,$$

and therefore

$$\frac{\phi_*(t+h) - \phi_*(t)}{h} = \frac{1}{h} \int_t^{t+h} F(\phi_*(s), s) ds \xrightarrow{h \rightarrow 0} F(\phi_*(t), t).$$

Since  $\phi_* \in \mathcal{A}$ , the latter limit is continuous, and thus the fixed point  $\phi_*$  of  $T$  is differentiable. Differentiation of the integral equation (1) with respect to  $t$  shows that

$$\phi_*'(t) = F(\phi_*(t), t).$$

Since  $\phi_*(t_0) = x_o$ , we see that  $\gamma_{(x_o, t_0)}(t) := \phi_*(t)$  is an integral curve of  $F$  through  $x_o$  at time  $t_0$ .

Conversely, any integral curve of  $F$  satisfies the equation (1), and is therefore a fixed point of  $T$ . Since contraction mappings have a unique fixed point, we see that any two integral curves must agree on  $I_\alpha$ . By carrying out the same proof in small intervals centered at all points of the intersection of the open set  $I_{(x,s)} \cap \tilde{I}_{(x,s)}$ , we obtain the uniqueness statement claimed in the theorem. The proof is therefore complete.  $\square$

#### 4. MAXIMAL INTEGRAL CURVES, FUNDAMENTAL DOMAINS, AND FLOWS

Our next goal is to ‘glue together’ the integral curves of a time-dependent vector fields. The first task is to maximally extend integral curves.

Let  $D \subset \mathbb{R}^n$  and  $V \subset D \times \mathbb{R}$  be domains, and let  $F : V \rightarrow \mathbb{R}^n$  be a continuous, time-dependent vector field such that for each  $s \in \mathbb{R}$ ,  $F_s : V_s \rightarrow \mathbb{R}^n$  is locally uniformly Lipschitz. Fix an initial condition  $(x, s) \in V$ . By Theorem 3.4,  $F$  has an integral curve through  $x$  with initial time  $s$ .

**PROPOSITION 4.1.** *With the notation above, there exists a unique integral curve  $\gamma_{(x,s)} : I_{(x,s)} \rightarrow D$  for  $F$  passing through  $x$  with initial time  $s$  such that if  $\phi : I \rightarrow D$  is another integral curve for  $F$  through  $(x, s)$  then  $I \subset I_{(x,s)}$ .*

*Proof.* With respect to inclusion of domains, the set  $\mathcal{I}_{(x,s)}$  of all integral curves for  $F$  passing through  $x$  with initial time  $s$  is partially ordered. Moreover, given two such integral curves  $\phi_i : I_i \rightarrow D$ ,  $i = 1, 2$ , Theorem 3.4 implies that the function

$$\phi(t) := \begin{cases} \phi_1(t), & t \in I_1 \\ \phi_2(t), & t \in I_2 \end{cases}$$

is well-defined, and therefore  $\phi : I_1 \cup I_2 \rightarrow D$  is also an integral curve for  $F$  passing through  $x$  with initial time  $s$ . It follows that  $\mathcal{I}_{(x,s)}$  is a directed set. We have to show that it has a maximal element, which is then of course unique.

To this end, let  $\{\phi_i : I_i \rightarrow D\}_{i \in I}$  be a maximal linearly ordered subset of  $\mathcal{I}_{(x,s)}$ . Then the set  $I := \bigcup_{i \in I} I_i$  is open, and the curve  $\phi : I \rightarrow D$  defined by

$$\phi(t) = \phi_i(t), \quad t \in I_i$$

is well-defined by the uniqueness part of Theorem 3.4, and therefore in  $\mathcal{I}_{(x,s)}$ . Thus  $\mathcal{I}_{(x,s)}$  has a unique maximal element in  $\mathcal{I}_{(x,s)}$ .  $\square$

DEFINITION 4.2. The unique maximal element of the set  $\mathcal{I}_{(x,s)}$  defined in the proof of the previous proposition is called the *maximal integral curve for  $F$  through  $(x, s)$* . We shall denote the maximal integral curve for  $F$  through  $(x, s)$  by

$$\Gamma_{(x,s)} : \mathcal{I}_{(x,s)} \rightarrow D.$$

One can also consider the unions of the graphs of the maximal integral curves.

DEFINITION 4.3. The set

$$\mathcal{U}_F := \{(x, s, t) ; (x, s) \in V, t \in \mathcal{I}_{(x,s)}\} \subset V \times \mathbb{R}$$

is called the fundamental domain of the time-dependent vector field  $F$ , and the map

$$\Phi_F : \mathcal{U}_F \rightarrow V$$

defined by  $\Phi_F(x, s, t) := (\Gamma_{(x,s)}(t), t)$  is called the time-dependent flow of  $F$ .  $\diamond$

Let us denote by  $\pi_D : V \rightarrow D$  the restriction to  $V$  of the natural projection  $\pi : D \times \mathbb{R} \rightarrow D$ .

DEFINITION 4.4. The map  $\Phi_s^t : D \rightarrow D$

$$(2) \quad \Phi_s^t(x) := \Gamma_{(x,s)}(t) = \pi_D \circ \Phi_F(x, s, t)$$

is called the time- $t$  map for the initial time  $s$ .  $\diamond$

The uniqueness part of Theorem 3.4 implies a symmetry appearing in the composition law for the maps (2), stated in the following result.

PROPOSITION 4.5. *For each  $s \in \mathbb{R}$  one has*

$$\Phi_s^s(x) = x \quad \text{for all } x \in V_s.$$

*Moreover, if  $(x, s, t) \in \mathcal{U}_F$  and  $(\Phi_s^t(x), t, r) \in \mathcal{U}_F$ , we have the pseudo-group law*

$$\Phi_t^r \circ \Phi_s^t(x) = \Phi_s^r(x).$$

## 5. AUTONOMOUS VECTOR FIELDS

From the point of view of classical mechanics, the general setting of time-dependent vector fields corresponds to physical systems in which the laws of physics change with time. Such situations can happen, but in nature we mostly find them when the particular physical system we are studying is not *closed*, i.e., it is part of a larger physical system.

By definition, the vector field representing a closed physical system is autonomous. That is to say, for each  $x \in D$

$$t \mapsto F(x, t)$$

is constant. In this case, we choose the convention of always taking initial value problems to start at time  $s = 0$ .

The fundamental domain and the flow are defined just slightly differently, so as to eliminate the initial time. Let us make the definitions precise.

DEFINITION 5.1. Let  $\xi : D \rightarrow \mathbb{R}^n$  be a vector field on a domain  $D \subset \mathbb{R}^n$ .

(i) The *maximal integral curve* for  $\xi$  through  $x \in D$  is the maximal integral curve

$$\Gamma_x : \mathcal{I}_x \rightarrow D$$

where  $\Gamma_x := \Gamma_{(x,0)}$  and  $\mathcal{I}_x := \mathcal{I}_{(x,0)}$ .

(ii) The *fundamental domain* of  $\xi$  is the domain

$$\mathcal{U}_\xi^0 := \{(\Gamma_x(t), t) ; x \in D\} \subset D \times \mathbb{R}.$$

(iii) The flow of  $\xi$  is the map  $\Phi_\xi : \mathcal{U}_\xi^0 \rightarrow D$  defined by

$$\Phi_\xi(x, t) = \Gamma_x(t).$$

The time- $t$  map is the map  $\Phi_\xi^t$  defined by

$$\Phi_\xi^t(x) = \Phi_\xi(x, t).$$

◇

Note that  $\mathcal{U}_\xi^0$  always contains  $D \times \{0\}$ . Note as well that the time- $t$  maps define the pseudo-group law

$$(3) \quad \Phi_\xi^t \circ \Phi_\xi^s = \Phi_\xi^{t+s}.$$

The link between the autonomous and time-dependent scenarios is the identity

$$\Phi_s^t = \Phi_\xi^{t-s}.$$

The pseudo-group law (3) is not a group law only because integral curves are not defined for a long enough time, i.e., even if  $t$  and  $s$  both lie in the domains of their respective integral curves,  $t+s$  may not. The situation in which this failure does not happen is therefore particularly important, and we study it in more detail now.

DEFINITION 5.2. A vector field  $\xi : D \rightarrow \mathbb{R}^n$  is said to be *complete* (sometimes also called *completely integrable*) if every maximal integral curve is defined on the entire real line. ◇

We have the following simple Proposition.

PROPOSITION 5.3. Let  $\xi : D \rightarrow \mathbb{R}^n$  be a locally Lipschitz vector field defined on a domain  $D \subset \mathbb{R}^n$ . Then the following are equivalent.

- (i)  $\xi$  is complete.
- (ii) There exists a positive number  $\varepsilon$  such that for each  $x \in D$ ,  $\mathcal{I}_x \supset (-\varepsilon, \varepsilon)$ .
- (iii) For each  $t \in \mathbb{R}$ , the map  $\Phi_\xi^t$  is a  $\mathcal{C}^1$ -diffeomorphism of  $D$ :  $\Phi_\xi^t \in \text{Diff}^1(D)$ .
- (iv) For some  $t \in \mathbb{R} - \{0\}$ ,  $\Phi_\xi^t \in \text{Diff}^1(D)$ .
- (v) The set of maps  $\{\Phi_\xi^t\}_{t \in \mathbb{R}}$  is a 1-parameter subgroup of  $\text{Diff}^1(D)$ .
- (vi) The fundamental domain of  $\xi$  is  $D \times \mathbb{R}$ .

The proof is left to the reader as an exercise.

## 6. APPROXIMATION

In this section we study a technique, initiated by Euler, for the approximation of integral curves and more generally flows. We confine ourselves to autonomous vector fields for the time being.

**DEFINITION 6.1.** Let  $\xi : D \rightarrow \mathbb{R}^n$  be a vector field on a domain  $D \subset \mathbb{R}^n$  and let  $I \subset \mathbb{R}$  be an open interval containing 0. An *algorithm* for  $\xi$  is a map  $H : D \times I \rightarrow D$  such that, with  $H_t(x) := H(x, t)$ ,

- (i)  $H_0 = \text{Id}$ ,
- (ii)  $H(x, \cdot)$  is  $\mathcal{C}^1$  and its derivative is continuous in  $D \times I$ , and
- (iii)  $\frac{\partial H}{\partial t} \Big|_{t=0} = \xi$ .

The basic approximation theorem is the following result.

**THEOREM 6.2.** *Let  $H$  be an algorithm for a Lipschitz vector field  $\xi$ . If  $(t, x) \in \mathcal{U}_\xi^0$  then for all  $N \gg 0$ ,  $H_{t/N}^{(N)}(x)$  is defined, and converges to  $\Phi_\xi^t(x)$ . Conversely, if  $H_{t/N}^{(N)}(x)$  is defined and converges for  $t \in [0, T]$  then  $(T, x) \in \mathcal{U}_\xi^0$  and*

$$\lim_{N \rightarrow \infty} H_{t/N}^{(N)}(x) = \Phi_\xi^t(x).$$

*In both statements, the convergence is locally uniform on  $D \times I$ .*

Before proving Theorem 6.2, we establish the following lemma which we shall need.

**LEMMA 6.3.** *Fix a Lipschitz vector field  $\xi : D \rightarrow \mathbb{R}^n$  defined on a domain  $D \subset \mathbb{R}^n$ , a point  $x_o \in D$  and a number  $\varepsilon \in (0, \text{dist}(x, D^c))$ . Fix a constant  $K > 0$  such that*

$$\|\xi(x) - \xi(y)\| \leq K\|x - y\|, \quad x, y \in B(x_o, \varepsilon).$$

*Then for any interval  $I \subset \mathbb{R}$  containing 0 such that  $\Phi_\xi^t(x)$  is defined for all  $t \in I$  and  $x \in B(x_o, \varepsilon)$ , we have the estimate*

$$\|\Phi_\xi^t(x) - \Phi_\xi^t(y)\| \leq e^{K|t|}\|x - y\|, \quad x, y \in B(x_o, \varepsilon), \quad t \in I.$$

*Proof.* Observe that with  $f(t) := \|\Phi_\xi^t(x) - \Phi_\xi^t(y)\|$  we have

$$f(t) = \left\| x - y + \int_0^t (\xi(\Phi_\xi^s(x)) - \xi(\Phi_\xi^s(y))) ds + x - y \right\| \leq \|x - y\| + K \int_0^t f(s) ds =: g(t).$$

Now,  $g'(t) = Kf(t) \leq Kg(t)$ , and we have

$$\frac{d}{dt} \left( e^{-Kt} g(t) \right) \leq 0.$$

Thus

$$g(t) \leq g(0)e^{Kt} \leq g(0)e^{K|t|}$$

which is what we want. □

*Proof of Theorem 6.2.* We begin by showing that the convergence holds locally. To this end, let  $x_o \in D$ . Then

$$(4) \quad H_t(x) = x + O(t) \quad \text{and} \quad \Phi_\xi^t(x) - H_t(x) = o(t).$$

If  $H_{t/j}^{(j)}(x)$  is well-defined for  $x$  in a small neighborhood of  $x_o$ , for  $j = 1, 2, \dots, N-1$ , then the semi-group law for time- $t$  maps and the first estimate in (4) shows that

$$\begin{aligned} H_{t/N}^{(N)}(x) - x &= H_{t/N}^{(N)}(x) - H_{t/N}^{(N-1)}(x) + H_{t/N}^{(N-1)}(x) - H_{t/N}^{(N-2)}(x) \\ &\quad + \dots + H_{t/N}(x) - x \\ &= NO(t/N) = O(t), \end{aligned}$$

which is small independently of  $N$ , for  $t$  sufficiently small. Thus for  $x$  sufficiently close to  $x_o$  and  $t$  sufficiently small,  $H_{t/N}^{(N)}(x)$  remains close to  $x_o$  for all  $N$ . In other words, with

$$x_j = H_{t/j}^{(j)}(x),$$

$\|x_j - x_o\| < \varepsilon$  for  $x$  sufficiently close to  $x_o$  and  $t$  sufficiently small. From the semi-group law for  $\Phi_\xi^t$ , we also have

$$\begin{aligned} \Phi_\xi^t(x) - H_{t/N}^{(N)}(x) &= (\Phi_\xi^{t/N})^{(N)}(x) - H_{t/N}^{(N)}(x) \\ &= (\Phi_\xi^{t/N})^{(N-1)}(\Phi_\xi^{t/N}(x)) - (\Phi_\xi^{t/N})^{(N-1)}(H_{t/N}(x)) \\ &\quad + \sum_{j=2}^N (\Phi_\xi^{t/N})^{(N-j)}(\Phi_\xi^{t/N}(x_j)) - (\Phi_\xi^{t/N})^{(N-j)}(H_{t/N}(x_j)), \end{aligned}$$

Thus, by repeated application of Lemma 6.3 we find the estimate

$$\begin{aligned} \|\Phi_\xi^t(x) - H_{t/N}^{(N)}(x)\| &\leq \sum_{k=1}^N e^{K|t|(N-k)/N} \|\Phi_\xi^{t/N}(x_{N-k-1}) - H_{t/N}(x_{N-k-1})\| \\ &\leq Ne^{K|t|} o(t/N), \end{aligned}$$

and the last quantity converges, as  $N \rightarrow \infty$ , to 0 uniformly on a small ball centered at  $x_o$  and for all sufficiently small  $t$ . The final estimate uses the second estimate of (4).

Having handled the case of short times, we now proceed to longer times. To this end, suppose first that  $\Phi_\xi^t(x)$  is defined for all  $t \in [0, T]$ . By what we have just done, if  $k$  is sufficiently large then

$$\Phi_\xi^{t/k}(y) = \lim_{k \rightarrow \infty} H_{t/k}^{(k)}(y)$$

holds uniformly for  $t \in [0, T]$  and  $y$  in a bounded neighborhood of the curve  $\{\Phi_\xi^t(x) ; t \in [0, T]\}$ . Thus

$$\Phi_\xi^t(x) = (\Phi_\xi^{t/k})^{(k)}(x) = \lim_{N \rightarrow \infty} (H_{t/(kN)}^{(N)})^{(k)}(x) = \lim_{N \rightarrow \infty} H_{t/(kN)}^{(Nk)}(x) = \lim_{N \rightarrow \infty} H_{t/N}^{(N)}(x).$$

Conversely, suppose  $t \mapsto H_{t/N}^{(N)}(x)$  converges to a curve  $c : [0, T] \rightarrow D$ . Let

$$S = \{t \in [0, T] ; \Phi_\xi^t(x) \text{ is defined and equal to } c(t)\}.$$

Clearly  $0 \in S$ , and from the local result  $S$  is relatively open. Let  $\{t_k\} \subset S$  and suppose  $t_k \rightarrow t$ . Then  $\Phi_\xi^{t_k}(x) \rightarrow c(t)$  so by Theorem 3.4  $\Phi_\xi^t(x)$  is defined, and by continuity,  $\Phi_\xi^t(x) = c(t)$ . Thus  $S$  is closed, and hence  $S = [0, T]$ .

Finally, observe that by existence and uniqueness,  $\Phi_\xi^{-t} = \Phi_{-\xi}^t$ , so the above proof applies to negative times as well.  $\square$



## 7. SUSPENSION: A REMARK

Autonomous vector fields are special cases of time-dependent vector fields. In this section, we note that the converse is true. To this end, let  $D \subset \mathbb{R}^n$  and  $V \subset D \times \mathbb{R}$  be domains and let  $F : V \rightarrow \mathbb{R}^n$  be a time-dependent vector field. Define  $\xi_F : V \rightarrow \mathbb{R}^n \times \mathbb{R}$  by the formula

$$\xi_F(x, s) := (F(x, s), 1)$$

The vector field  $\xi_F$  is then autonomous, and its flow is given by the time- $t$  maps

$$\Phi_{\xi_F}^t(x, s) = (\Phi_s^{s+t}(x), s + t).$$

It is therefore possible to extract the flow of  $F$  from that of  $\xi_F$ . If one can find the latter flow, this is of course possible. In fact, the hypotheses of Theorem 3.4 apply to  $\xi_F$  as soon as  $F$  is Lipschitz on  $V$ .