The $\frac{1}{3}$-Trick

Let

$$\Delta_{\mathbb{R}} := \{[\frac{i}{2^k}, \frac{i+1}{2^k}) : \; i, k \in \mathbb{Z}\}$$

and for an interval I, let

$$\Delta_I := \{[a,b) \in \Delta_{\mathbb{R}} : \; [a,b) \subset I\} \, .$$

Similarly, let

$$\Delta_{\mathbb{R}^n} := \{I_1 \times ... \times I_n : I_j \in \Delta_{\mathbb{R}}; \; |I_j| = |I_\ell|\}$$

and

$$\Delta_Q := \{R \in \Delta_{\mathbb{R}^n} : R \subset Q\}$$

For $U \subset \mathbb{R}^n$, and $v \in \mathbb{R}^n$ denote by $U + v$ the set

$$U + v := \{u + v : u \in U\}.$$

If $\mathcal{U}$ is a collection of subsets of $\mathbb{R}^n$, then we denote by $\mathcal{U}^{+v}$ the collection

$$\mathcal{U}^{+v} := \{U + v : U \in \mathcal{U}\} \, .$$

The following are 3 pretty easy questions. The main lesson is "any ball in $\mathbb{R}^n$ is contained in a cube of roughly the same size, where the cube is from one of several dyadic grids". Let $J_0 = [-1, 2)$ and $Q_0 = [-1, 2)^n \subset \mathbb{R}^n$.

1. Show that there is an interval $J \subset [0, 1)$ such that $J \notin \Delta_{J_0}$

2. Show that there is a constant $C > 0$ such that for any $x \in [0, 1)$, and $r \in (0, \frac{1}{C})$ we have an interval J such that $(x - r, x + r) \subset J$, and

   $$J \in \Delta_{J_0} \cup \Delta_{J_0}^{+\frac{1}{3}}$$

   where J satisfies $r \leq |J| < Cr$.

   Hint: you will need to use the fact that 3 and 2 are co-prime. It could be useful to write $\frac{1}{3}$ in binary.

3. Conclude that there is a constant $C_n > 0$ and vectors $v_1, v_2, ..., v_N \in \mathbb{R}^n$, where $N = 2^n$, such that for any $x \in [0,1)^n$, and $r \in (0, \frac{1}{C_n})$ we have a cube Q such that $B(x,r) \subset Q$, and

$$Q \in \bigcup_{i=1}^{N} \Delta_{Q_0}^{v_i}$$

where Q satisfies $r \leq \operatorname{diam}(Q) < Cr$.

Hint: take $v_i \in \{0, \frac{1}{3}\}^n$.