1. (20 points) Calculate the following antiderivatives.

(i)

$$\int \frac{1}{x^2-4x} dt.$$

Answer: Rewrite the denominator as $x(x-4)$ and use a partial fractions decomposition.

$$\frac{1}{x^2-4x} = \frac{A}{x} + \frac{B}{x-4}$$
$$1 = A(x-4) + Bx$$
$$1 = (A+B)x - 4A$$

By matching coefficients, we obtain the equations $A+B=0, -4A = 1$. The solutions are $A = -1/4, B = 1/4$. Then

$$\int \frac{1}{x^2-4x} dx = \int \frac{-1/4}{x} + \frac{1/4}{x-4} dx$$
$$= \boxed{-\frac{1}{4}\ln|x| + \frac{1}{4}\ln|x-4| + C.}$$

(ii)

$$\int \frac{8x-7}{x^2-4x+3} dt.$$

Answer: Factor the denominator as $x^2-4x+3 = (x-1)(x-3)$, then

$$\frac{8x-7}{x^2-4x+3} = \frac{A}{x-1} + \frac{B}{x-3}$$
$$8x-7 = A(x-3) + B(x-1)$$
$$8x-7 = (A+B)x + (-3A-B).$$

We get the equations $A+B=8, -3A-B=-7$. These are solved by $A=-1/2, B=17/2$. Then

$$\int \frac{8x-7}{x^2-4x+3} dx = \int \frac{-1/2}{x-1} + \frac{17/2}{x-3} dx$$
$$= \boxed{-\frac{1}{2}\ln|x-1| + \frac{17}{2}\ln|x-3| + C.}$$

2. (30 points) Show the divergence or convergence of the following integrals.

(i)

$$\int_0^\infty \frac{8x}{x^2+1}dx$$

Answer: We show divergence by directly evaluating the integral, using the u-substitution $u = x^2 + 1$. Then $du = 2dx$, and

$$\begin{aligned}
\int_0^\infty \frac{8x}{x^2+1}\,dx &= \lim_{t\to\infty} \int_0^t \frac{8x}{x^2+1}\,dx \\
&= \lim_{t\to\infty} \int_1^{t^2+1} \frac{4}{u}\,du \\
&= \lim_{t\to\infty} \left[4\ln|u|\right]_1^{t^2+1} \\
&= \lim_{t\to\infty} \left[4\ln|t^2+1| - 4\ln|1|\right] \\
&= +\infty.
\end{aligned}$$

This shows that the integral diverges.

(ii)

$$\int_0^\infty \frac{1}{x^{45}+5}dx$$

Answer: We show convergence by the comparison test. Note that we have that for $x > 1$:

$$0 < \frac{1}{x^{45}+5} < \frac{1}{x^{45}},$$

and $\int_1^\infty \frac{1}{x^{45}}\,dx$ converges. Then

$$\int_0^\infty \frac{1}{x^{45}+5}\,dx = \int_0^1 \frac{1}{x^{45}+5}\,dx + \int_1^\infty \frac{1}{x^{45}+5}\,dx$$

is a sum of a proper integral (which automatically converges) and an improper integral, which converges by comparison with the function $\frac{1}{x^{45}}$. Therefore the given integral converges.

(iii)

$$\int_0^\infty \frac{1}{x^2-1}dx$$

Answer: We show divergence by direct evaluation. The integrand has a pole at $x = 1$, so we first split the improper integral into two improper integrals:

$$\int_0^\infty \frac{1}{x^2-1}\,dx = \int_0^1 \frac{1}{x^2-1}\,dx + \int_1^\infty \frac{1}{x^2-1}\,dx.$$

It is enough to prove that one of these integrals diverges. Consider the first. By factoring the denominator $x^2 - 1 = (x-1)(x+1)$, we have the partial fractions decomposition

$$\begin{aligned}\frac{1}{x^2-1} &= \frac{A}{x-1} + \frac{B}{x+1}\\ 1 &= A(x+1) + B(x-1)\\ 1 &= (A+B)x + (A-B).\end{aligned}$$

The equations $A + B = 0, A - B = 1$ has solutions $A = 1/2, B = -1/2$. Then

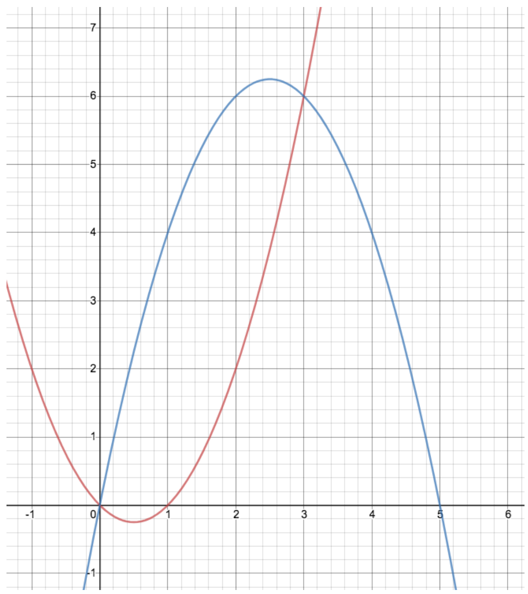
$$\begin{aligned}\int_0^1 \frac{1}{x^2-1}\,dx &= \lim_{a\to 1^-}\int_0^a \frac{1}{x^2-1}\,dx\\ &= \lim_{a\to 1^-}\int_0^a \frac{1/2}{x-1} - \frac{1/2}{x+1}\,dx\\ &= \lim_{a\to 1^-}\left[\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right]_0^a\\ &= \lim_{a\to 1^-}\left[\frac{1}{2}\ln|a-1| - \frac{1}{2}\ln|a+1| - \frac{1}{2}\ln|0-1| + \frac{1}{2}\ln|0+1|\right]\\ &= -\infty,\end{aligned}$$

since ln has a pole at 0. This proves the integral diverges.

3. (30 points)

(i) Sketch the region bounded by $y = x^2 - x$ and $y = 5x - x^2$

Answer:



(ii) Find the area of the region above

Answer: First we need to determine where the curves intersect. This is done by setting equal the two equations and solving for x.

$$x^2 - x = 5x - x^2$$
$$2x^2 - 6x = 0$$
$$x(x-3) = 0$$

This shows that the curves intersect at $x = 0, 3$. From the picture above, the curve $5x - x^2$ is on top, so the area of the enclosed region is

$$\int_0^3 (5x - x^2) - (x^2 - x)\,dx = \int_0^3 6x - 2x^2\,dx$$
$$= \left[\frac{6}{2}x^2 - \frac{2}{3}x^3\right]_0^3$$
$$= \boxed{9.}$$

(iii) The birth rate of a population is

$$b(t) = 4000e^{0.5t}$$

per year, and the death rate is

$$d(t) = 1700e^{0.4t}$$

per year. Find the area between these curves and the times $t = 0$ and $t = 10$ years. What does this area represent?

Answer: The area between the curves is

$$\int_0^{10} b(t) - d(t)\,dt = \int_0^{10} 4000e^{0.5t} - 1700e^{0.4t}\,dt$$
$$= \left[\frac{4000}{0.5}e^{0.5t} - \frac{1700}{0.4}e^{0.4t}\right]_0^{10}$$
$$= (8000e^5 - 4250e^4) - (8000 - 4250)$$
$$= \boxed{8000e^5 - 4250e^4 - 3750.}$$

The area represents the net change in population over 10 years.

4. (20 points) Determine the following antiderivatives.

(i)

$$\int \sin^2(3x)dx$$

Answer: Using the identity $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$, we get

$$\int \sin^2(3x)\,dx = \int \frac{1}{2}(1-\cos(6x))\,dx$$
$$= \boxed{\frac{x}{2} - \frac{1}{12}\sin(6x) + C.}$$

(ii)

$$\int \sin^3(7x)\cos^2(7x)dx$$

Answer: Using the identity $\sin^2(x) = 1-\cos^2$, the integrand becomes

$$\sin^3(7x)\cos^2(7x) = \sin(7x)(1-\cos^2(7x))\cos^2(7x).$$

Then with the substitution $u = \cos(7x)$, we have $du = -7\sin(7x)dx$ and

$$\int \sin^3(7x)\cos^2(7x)\,dx = \int \sin(7x)(1-\cos^2(7x))\cos^2(7x)\,dx$$
$$= \int -\frac{1}{7}(1-u^2)u^2\,du$$
$$= -\frac{1}{7}(\frac{u^3}{3} - \frac{u^5}{5}) + C$$
$$= \boxed{\frac{1}{35}\cos^5(7x) - \frac{1}{21}\cos^3(7x) + C.}$$

5. (20 points). Use a trigonometric substitution to determine the following antiderivatives.

(i)

$$\int \frac{3}{x^2+1} dx$$

Answer: Set $x = \tan\theta$, so $dx = \sec^2\theta \, d\theta$. Then

$$\begin{aligned}
\int \frac{3}{x^2+1} \, dx &= \int \frac{3\sec^2\theta}{\tan^2 + 1} \, d\theta \\
&= \int \frac{3\sec^2\theta}{\sec^2\theta} \, d\theta \\
&= 3\theta + C \\
&= \boxed{3\arctan(x) + C.}
\end{aligned}$$

(ii)

$$\int \frac{4}{\sqrt{1-x^2}} dx$$

Answer: Set $x = \sin\theta$, so $ds = \cos\theta \, d\theta$. Then

$$\begin{aligned}
\int \frac{4}{\sqrt{1-x^2}} \, dx &= \int \frac{4\cos\theta}{\sqrt{1-\sin^2\theta}} \, dx \\
&= \int \frac{4\cos\theta}{\cos\theta} \, dx \\
&= 4\theta + C \\
&= \boxed{4\arcsin(x) + C.}
\end{aligned}$$