1. (20 points)

(i) Calculate the distance between the point $(1, -2, 3)$ and the point $(2, 7, 1)$.

(ii) Find the equation of the line going through $(2, -1, -1)$ and $(1, 2, -3)$.

(iii) Do the points $(2, -2, -1)$, $(1, 2, -3)$ and $(0, 9, -5)$ lie on a straight line?

i) Distance $= \sqrt{(1-2)^2 + (-2-7)^2 + (3-1)^2} = \sqrt{86}$

ii) A vector going from $(2,-1,-1)$ to $(1,2,-3)$ is given by $\langle 1-2, 2-(-1), -3-(-1) \rangle = \langle -1, 3, -2 \rangle$

So the parametric equations are

$$x = 2 - t$$
$$y = -1 + 3t$$
$$z = -1 - 2t$$

iii)

If $(0,9,-5)$ lies on the straight line going through $(2,-1,-1)$ and $(1,2,-3)$,

then by part ii)

$$0 = 2 - t$$
$$9 = -1 + 3t$$
$$-5 = -1 - 2t$$

So $t = 2$, however $9 \neq -1 + 3(2)$,

Therefore the three points do not lie on a straight line.

2. (20 points) Find the parametric equation of the tangent line to the curve

$$\vec{r}(t) = \langle 4t,\ \sqrt{2}\cos(t),\ \sqrt{2}\sin(t)\rangle$$

at the point $(\pi, 1, 1)$.

$(\pi, 1, 1)$ corresponds to $t = \frac{\pi}{4}$

$\vec{r}\,'(t) = \langle 4, -\sqrt{2}\sin t, \sqrt{2}\cos t\rangle$

$\vec{r}\,'(\frac{\pi}{4}) = \langle 4, -1, 1\rangle$

So the parametric equations of the tangent line are

$x = \pi + 4t$

$y = 1 - t$

$z = 1 + t$

3. Consider the points $P = (0,0,3)$, $Q = (0,1,0)$ and $R = (2,0,0)$. There is a plane going through them.

(i) Find a normal vector to this plane.

(ii) Find a scalar equation for this plane.

i) $\overrightarrow{PQ} = \langle 0, 1, -3 \rangle$

$\overrightarrow{PR} = \langle 2, 0, -3 \rangle$

A normal vector to this plane is $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & -3 \\ 2 & 0 & -3 \end{vmatrix}$

$= -3\underline{i} - 6\underline{j} - 2\underline{k}$

ii) A scaler equation for this plane is

$-3x - 6y - 2z + d = 0$ for some $d$.

Since $P$ lies on this plane,

$-3(0) - 6(0) - 2(3) + d = 0 \Rightarrow d = 6$.

Therefore we get

$-3x - 6y - 2z + 6 = 0$

4. (20 points)

(i) Find the point of intersection beween the two curves

$$\begin{aligned}\vec{r}_1(t) &= \langle t^2, t^3, t\rangle \\ \vec{r}_2(s) &= \langle s-2, 3s-8, s-2\rangle\end{aligned}$$

(ii) What is $\cos(\alpha)$, where $\alpha$ is the angle between the curves at the point of intersection?

i) From $\vec{r}_1(t) = \vec{r}_2(s)$ we get

$t^2 = s-2$

$t^3 = 3s-8$

$t = s-2$

so $t^2 = s-2 = t$, $\quad t(t-1) = 0$, $\quad t = 0, 1$

For $t=0$, $s=2$, but $(0)^3 \neq 3(2)-8$

For $t=1$, $s=3$, and $(1)^3 = 3(3)-8$

Therefore the point of intersection is $\vec{r}_1(1) = \vec{r}_2(3) = \langle 1,1,1\rangle$

ii) $\vec{r}_1'(t) = \langle 2t, 3t^2, 1\rangle$, $\quad \vec{r}_1'(1) = \langle 2,3,1\rangle$

$\vec{r}_2'(s) = \langle 1, 3, 1\rangle$, $\quad \vec{r}_2'(3) = \langle 1,3,1\rangle$

$$\cos\alpha = \frac{\langle 2,3,1\rangle \cdot \langle 1,3,1\rangle}{|\langle 2,3,1\rangle|\ |\langle 1,3,1\rangle} = \frac{12}{\sqrt{154}}$$

5. (20 points) What is the arc-length of

$$\vec{r}(t) = \langle 4t, 2\sin(t), 2\cos(t)\rangle, \quad 0 \le t \le \pi$$

$$r'(t) = \langle 4, 2\cos(t), -2\sin(t)\rangle$$

$$|r'(t)| = \sqrt{4^2 + (2\cos(t))^2 + (-2\sin(t))^2}$$

$$= \sqrt{16 + 4(\cos^2(t) + \sin^2(t))}$$

$$= \sqrt{16 + 4}$$

$$= \sqrt{20}$$

$$L = \int_0^\pi |r'(t)|\,dt = \int_0^\pi \sqrt{20}\,dt = \sqrt{20}\,t\Big|_0^\pi = \sqrt{20}\,\pi$$

$$= 2\sqrt{5}\,\pi$$

Common mistakes:

$$L \neq \int_0^\pi r(t)\,dt$$

$$L \neq \int_0^\pi |r(t)|\,dt$$

$$L \neq \int |4|\,dt + \int |2\cos(t)|\,dt + \int |2\sin(t)|\,dt$$

6. (20 points) Find the position function of a particle that has an initial velocity $\vec{v}(0) = \langle 1, 1, 0 \rangle$ and has acceleration $\vec{a}(t) = \langle t, t^2, \sin(t) \rangle$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt \quad + \text{ correct int'l cdn}$$

$$= \left\langle \tfrac{1}{2}t^2, \tfrac{1}{3}t^3, -\cos t \right\rangle + \text{ in. cdn.}$$

I want $\vec{v}(0) = \langle 1, 1, 0 \rangle$. As it stands,

$\vec{v}(0) = \langle 0, 0, -1 \rangle$, so

$$\vec{v}(t) = \left\langle \tfrac{1}{2}t^2 + 1, \tfrac{1}{3}t^3 + 1, -\cos t + 1 \right\rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt \quad + \text{ any int'l cdn}$$

$$= \left\langle \tfrac{1}{6}t^3 + t, \tfrac{1}{12}t^4 + t, -\sin t + t \right\rangle + \vec{r}_0$$

where $\vec{r}_0$ is any vector in $\mathbb{R}^3$.