Description of Publications.

Raanan Schul

All papers available on my home page:
`http://www.math.sunysb.edu/~schul`
For up to date citation counts see Google Scholar:
`http://scholar.google.com/citations?&user=wYsAdQsAAAAJ`

The following is a description of all the papers I have authored or coauthored. It is arranged chronologically. In most cases, I chose to cut and paste the abstract or parts of the introduction of the paper.

1. R. Schul. Subsets of rectifiable curves in Hilbert space. *Journal d'Analyse Mathématique* 103 (2007), 331-375. (This paper is essentially the same as my PhD thesis).

   The aim of this paper is to classify subsets of Hilbert spaces that are contained in a connected set of finite Hausdorff length. We do so by extending and improving results of Peter Jones and Kate Okikiolu for sets in $\mathbb{R}^d$. We prove a quantitative version of the following statement: a connected set of finite Hausdorff length (or a subset of one), is characterized by the fact that inside balls at most scales around most points of the set, the set lies close to a straight line segment (which depends on the ball). More precisely, define for any set E in a Hilbert space

$$\beta_{\infty,E}(Q) \quad = \quad \frac{1}{\mathrm{diam}(Q)} \inf_{L \text{ line}} \sup_{x\in E\cap Q} \mathrm{dist}(x,L)\,.$$

   Assume $\mathrm{diam}(E) = 1$. Let $A > 1$ be fixed. Set $X_{-1} = \emptyset$ and for $n \geq 0$ let $X_n \subset X_{n+1} \subset E$ be a 2^{-n} net. Let

$$\mathcal{G}^E = \{\mathrm{Ball}(x, A2^{-n}) :\ x \in X_n,\ n \geq 0\}\,. \tag{1}$$

   Theorem 1. *For any connected set $\Gamma \subset \mathbb{R}^d$ or $\Gamma \subset H$ (Hilbert space) and any $E \subset \Gamma$ we have*

$$\sum_{\mathcal{G}^E} \beta^2_{\infty,\Gamma}(B)\mathrm{radius}(B) \lesssim \mathcal{H}^1(\Gamma). \tag{2}$$

   This was first proved for $d = 2$ by Jones in his 1990 paper *Rectifiable sets and the traveling salesman problem* using complex analysis, and then extended to all $d < \infty$ by Okikiolu her 1992 paper *Characterization of subsets of rectifiable curves in* $\mathbb{R}^n$, using geometric methods. The constant that comes out of Okikiolu's proof depends exponentially on the dimension d

($= n$ is her notation). In our paper, the Hilbert space version of this theorem is proved, and in particular it is shown that the constants are independent of the dimension d. The following theorem, also proved in this paper, gives a very good reason to care about the left hand side of inequality (2).

Theorem 2. *Suppose A is large enough. Given a set $E \subset \mathbb{R}^d$ or $E \subset H$ (Hilbert space), there exists a connected set $\Gamma_0 \supset E$ such that the length of Γ_0 satisfies*

$$\mathcal{H}^1(\Gamma_0) \lesssim \operatorname{diam}(E) + \sum_{\mathcal{G}^E} \beta^2_{\infty,E}(B)\operatorname{radius}(B). \tag{3}$$

This was proved for $d < \infty$ by Jones' 1990 paper. Again, the constant which comes out of the original proof is exponential in the dimension d and in our paper the Hilbert space version of this theorem is proved. In particular it is shown that the constants are independent of the dimension d. As a corollary of Theorem 1 and Theorem 2, one gets is the following characterization of subsets of connected finite Hausdorff-length sets.

Corollary 3. *Suppose A_0 is large enough. Given a set $E \subset \mathbb{R}^d$ or $E \subset H$ (Hilbert space) the smallest Hausdorff-length of a connected set containing E is comparable, up to universal constants, to*

$$\operatorname{diam}(E) + \sum_{\mathcal{G}^E} \beta^2_{\infty,E}(B)\operatorname{radius}(B)\,.$$

(In particular, this length is also comparable to the length of Γ_0 constructed in Theorem 2.)

2. R. Schul. Analyst's traveling salesman theorems. A survey. *In the tradition of Ahlfors and Bers, IV*, volume 432 of *Contemp. Math.*, pages 209–220. Amer. Math. Soc., Providence, RI, 2007.

   Around the same time that Theorem 1 and Theorem 2 appeared, there were two versions of Theorem 2 by I. Hahlomaa for metric spaces (one appearing in 2005 and the other in 2007), and one version of Theorem 2 by Ferrari, Franchi, and Pajot for the Heisenberg group (appearing in 2007). The purpose of this survey was threefold. The first purpose was to group these results together with some older theorems in an expository manner. The second was to provide an example which showed that the analogue of Theorem 1 was false in Hahlomaa's 2005 metric space setting. Finally, a result which was the analogue of Theorem 1 corresponding to Hahlomaa's 2007 metric space version of Theorem 2 was announced. This result was shown in full in the paper below.

3. R. Schul. Ahlfors-regular curves in metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 32 (2007), 437-460.

   In this paper we give a metric space analogue of Theorem 1. We discuss 1-Ahlfors-regular connected sets in a general metric space and prove that such sets are 'flat' on most scales and in most locations. Our result is quantitative, and when combined with work of I. Hahlomaa (from 2007), gives a characterization of 1-Ahlfors regular subsets of 1-Ahlfors-regular curves in metric spaces. Our result is a generalization to the metric space setting of the Analyst's (Geometric) Traveling Salesman theorems of P. Jones, K. Okikiolu, and G. David and S. Semmes, and it can be stated in terms of average Menger curvature. In particular, Define $\beta_2(B)$ by

$$\beta_2^2(B)\mathrm{radius}(B) = \iiint_{(B\cap\Gamma)^3} \partial(\{x,y,z\})\mathrm{radius}(B)^{-3} d\mathcal{H}^1(z)d\mathcal{H}^1(y)d\mathcal{H}^1(x). \tag{4}$$

where

$$\partial(\{x_1,x_2,x_3\}) = \min_{\sigma\in S_3}\left(\mathrm{dist}(x_{\sigma(1)},x_{\sigma(2)}) + \mathrm{dist}(x_{\sigma(2)},x_{\sigma(3)}) - \mathrm{dist}(x_{\sigma(1)},x_{\sigma(3)})\right).$$

We show the following.

Theorem 4. *Let $\Gamma\subset\mathcal{M}$ be a connected 1-Ahlfors-regular set in a metric space. Let $E\subset\Gamma$ and let $\mathcal{G}^E$ be a multiresolution family as in equation (1). Then we have*

$$\sum_{B\in\mathcal{G}^E}\beta_2^2(B)\mathrm{radius}(B) \lesssim \mathcal{H}^1(\Gamma). \tag{5}$$

4. R. Schul, *Bi-Lipschitz decomposition of Lipschitz functions into a metric space*, Rev. Mat. Iberoam. **25** (2009), no. 2, 521–531.

   We prove a quantitative version of the following statement. Given a Lipschitz function f from the k-dimensional unit cube into a general metric space, one can be decomposed f into a finite number of bi-Lipschitz functions $f|_{F_i}$ so that the k-Hausdorff content of $f([0,1]^k \setminus \cup F_i)$ is small. We thus generalize a theorem of P. Jones (from 1988) from the setting of $\mathbb{R}^d$ to the setting of a general metric space. This positively answers problem 11.13 in *Fractured Fractals and Broken Dreams* by G. David and S. Semmes, or equivalently, question 9 from *Thirty-three yes or no questions about mappings, measures, and metrics* by J. Heinonen and S. Semmes. Our statements extend to the case of *coarse* Lipschitz functions. More precisely, we prove the following theorem.

   Theorem 5. *Let $\epsilon\geq 0$, $0<\alpha<1$ and $k\geq 1$ be given. There are universal constants $M=M(\alpha,k)$, $c_1=c_1(k)$ and c_2 such that the following statements hold. Let $\mathcal{M}$ be any metric space. Let $f:[0,1]^k\to\mathcal{M}$ be an ϵ-coarse 1-Lipschitz function, i.e. such that for all $x,y\in[0,1]^k$*

$$\mathrm{dist}(f(x),f(y)) \leq |x-y|+\epsilon.$$

   Then there are sets $F_1,...,F_M\subset[0,1]^k$ so that for $1\leq i\leq M$, $x,y\in F_i$ we have

$$\alpha|x-y| - c_2\epsilon \leq \mathrm{dist}(f(x),f(y)) \leq |x-y|+\epsilon,$$

   and

$$h^k(f([0,1]^k\setminus(F_1\cup...\cup F_M))) \leq c_1\alpha. \tag{6}$$

   (h^k is the one-dimensional Hausdorff content).

As a corollary, one gets that if a k-Ahlfors regular set in a metric space has Big Pieces of Lipschitz Images (BPLI) then it also has Big Pieces of BiLipschitz images (BPBI). This was previously only known for Euclidean subsets.

Remark 6. *We remark that Jones' proof relied on a multi-scale decomposition of ∇f, which can be rephrased as taking a wavelet expansion of ∇f. This made sense as the target was Euclidean. Our target space is a general metric space, and so we had to substitute this with an appropriate type of multi scale analysis. It relied on a notion of curvature, which close to Menger curvature (metric analogue). Menger curvature is a quantity that is related to β numbers. Thus, this is an example where the philosophy of 'β numbers are geometric analogues of wavelet coefficients' works beautifully. This philosophy was a guide in every case below where we were able to prove something about functions with a metric space image.*

5. R. Schul. Big-Pieces-of-Lipschitz-Images implies a sufficient Carleson estimate in a metric space. `arXiv:0706.2517`.

   This note is intended to be a supplement to the bi-Lipschitz decomposition of Lipschitz maps Theorem above. We show that in the case of 1-Ahlfors regular sets, the condition of having 'Big Pieces of bi-Lipschitz Images' (BPBI) is equivalent to a Carleson condition. The main point is that one can use a John-Nirenberg-Strömberg type Lemma to go from 'bi-Lipschitz Images' to 'Big Pieces of Bi-Lipschitz Images'. (As of 2014, it has never been submitted for publication...)

6. P. W. Jones, M. Maggioni, and R. Schul. Manifold parameterizations by eigenfunctions of the Laplacian and heat kernels. *Proc. Natl. Acad. Sci. USA*, 105(6):1803–1808, 2008.

   This paper served as an announcement to the general scientific community of the results which later appeared in full in the paper below. These results are of interest to a wide range of people who do analysis of data sets, such as people in the machine learning community and the electrical engineering community. (This is a very well cited paper)

7. P.W. Jones, M. Maggioni, and R. Schul. *Universal local parametrizations via heat kernels and eigenfunctions of the Laplacian*, Ann. Acad. Sci. Fenn. Math. **35** (2010), no. 1, 131–174.

   We use heat kernels or eigenfunctions of the Laplacian to construct local coordinates on large classes of Euclidean domains and Riemannian manifolds (not necessarily smooth, e.g. with $\mathcal{C}^\alpha$ metric). These coordinates are bi-Lipschitz on embedded balls of the domain or manifold, with distortion constants that depend only on natural geometric properties of the domain or manifold. The proof of these results relies on estimates, from above and below, for the heat kernel and its gradient, as well as for the eigenfunctions of the Laplacian and their gradient. These estimates hold in the non-smooth category, and are stable with respect to perturbations within this category. Finally, these coordinate systems are intrinsic and efficiently computable, and are of value in applications.

   For brevity, we make a precise statement here only of our theorem about Laplacian eigenfunctions of Euclidean domains.

Theorem 7 (Embedding via Eigenfunctions, for Euclidean domains)**.** *Let Ω be a finite volume domain in $\mathbb{R}^d$, rescaled so that* $\mathrm{volume}(\Omega) = 1$*. Let Δ be the Laplacian in Ω, with Dirichlet or Neumann boundary conditions, defined to have positive spectrum. We assume either Dirichlet or Neumann boundary conditions on Ω. We will assume that the spectrum of the Laplace operator is discrete, and denote by $0 \leq \lambda_0 \leq \cdots \leq \lambda_j$ its eigenvalues and by $\{\varphi_j\}$ the corresponding orthonormal basis of eigenfunctions. We will also assume*

$$\#\{j : 0 < \lambda_j \leq T\} \leq C_{count} T^{\frac{d}{2}} \mathrm{volume}(\Omega)\,. \tag{7}$$

Then there is a constant $\kappa > 1$ that depends only on d such that the following hold.

For any $z \in \Omega$, let $\rho \leq \mathrm{dist}\,(z, \partial\Omega)$. Then there exist integers $i_1, \ldots, i_d$ such that, if we let

$$\gamma_l = \left(\fint_{\mathrm{Ball}(z,\kappa^{-1}\rho)} \varphi_{i_l}^2 \right)^{-\frac{1}{2}} \,,\ l = 1, \ldots, d\,,$$

we have that:

(a) the map

$$\begin{aligned} \Phi : \mathrm{Ball}(z, \kappa^{-1}\rho) &\to \mathbb{R}^d && (8)\\ x &\mapsto (\gamma_1 \varphi_{i_1}(x), \ldots, \gamma_d \varphi_{i_d}(x)) && (9) \end{aligned}$$

satisfies, for any $x_1, x_2 \in \mathrm{Ball}(z, \kappa^{-1}\rho)$,

$$\frac{\kappa^{-1}}{\rho} ||x_1 - x_2|| \leq ||\Phi(x_1) - \Phi(x_2)|| \leq \frac{\kappa}{\rho} ||x_1 - x_2||\,; \tag{10}$$

(b) the associated eigenvalues satisfy

$$\kappa^{-1}\rho^{-2} \leq \lambda_{i_1}, \ldots, \lambda_{i_d} \leq \kappa\rho^{-2}\,;$$

(c) the constants γ_l satisfy

$$\gamma_1, \ldots, \gamma_d \leq \kappa\,(C_{count})^{\frac{1}{2}}.$$

8. C. Sormani and S. Wenger Weak Convergence and Cancellation. Appendix by R. Schul and S. Wenger. *Calc. Var. and PDE.* Vol. 38. Issue 1 (2010).

Sormani and Wenger study the relationship between the weak limit of a sequence of integral currents in a metric space and the possible Hausdorff limit of the sequence of supports. Due to cancellation, the weak limit is in general supported in a strict subset of the Hausdorff limit. They exhibit sufficient conditions in terms of topology of the supports which ensure that no cancellation occurs and that the support of the weak limit agrees with the Hausdorff limit of the supports. They use their results to prove countable $\mathcal{H}^m$-rectifiability of the Gromov-Hausdorff limit of sequences of Lipschitz manifolds M_n all of which are λ-linearly locally contractible up to some scale r_0. In the appendix, written by R. Schul and S. Wenger, it is shown that the Gromov-Hausdorff limit need not be countably $\mathcal{H}^m$-rectifiable if the M_n have a common local geometric contractibility function which is only concave (and not linear).

9. J. B. Garnett, R. Killip, R. Schul A doubling measure can charge a rectifiable curve *Proc. Amer. Math. Soc.*, 138(5):1673–1679, 2010.

   For $d \geq 2$, we construct a doubling measure ν on $\mathbb{R}^d$ and a rectifiable curve Γ such that $\nu(\Gamma) > 0$. In fact, for any $\epsilon > 0$, one has a rectifiable curve Γ such that $\nu(\mathbb{R}^d \setminus \Gamma) < \epsilon$. Note that the fact that ν is doubling, implies that $\nu(G) = 0$ whenever G is a Lipschitz graph, i.e. is the image of a function the form $x \mapsto A(x, g(x))$ where g is Lipschitz and A is affine.

   The existence of such a combination of doubling measure and rectifiable curve was previously unknown and was a question which was popular among the geometric function theory people decending from the University of Michigan and Finland. There are obvious questions that arise from this: Which measures ν have the above property? What if a curve is replaced by a Lipschitz surface?

10. J. Azzam and R. Schul, *How to take shortcuts in Euclidean space: making a given set into a short quasi-convex set*, Proc. Lond. Math. Soc. (3) **105** (2012), no. 2, 367–392.

    Theorem 8. *Let $d \geq 2$. There exist constants $C_1, C_2 > 1$, C_1 depending on d, such that for any subset $K \subset \mathbb{R}^d$ there exists a connected set $\widehat{\Gamma} \subset \mathbb{R}^d$ such that:*

    (i) $\widehat{\Gamma} \supset K$.

    (ii) $\mathcal{H}^1(\widehat{\Gamma}) \leq C_1 \mathcal{H}^1(\Gamma)$ *for any connected* $\Gamma \supset K$.

    (iii) For any $x, y \in \widehat{\Gamma}$ there is a path $\gamma_{x,y} \subset \widehat{\Gamma}$ connecting x and y, such that $\ell(\gamma_{x,y}) \leq C_2|x-y|$.

    This theorem has a nice description in layman's terms. Suppose you have a connected system of (one dimensional) roads in d-dimensional Euclidean space. Call this road system G. Loosely speaking, we showed that you can, relatively cheaply, extend this road system to be an efficient road system. Let us try to be a bit more precise. Suppose the total length of G is L, where by length we mean Hausdorff length. Between any two points x, y on the road-system there are now two distances one can measure. One, by a straight Euclidean line through space; call it $d_E(x, y)$. The other, along the road system; call it $d_G(x, y)$. Clearly, the direct distance is no larger that the one along G, i.e. $d_E(x, y) \leq d_G(x, y)$. It is easy to draw examples where the distance along the road system is much larger than the one through space. We showed in our paper that one can extend G to a new road system $\hat{G}$, which has total length no more than $C \cdot L$. This new road system, $\hat{G}$ will have the property that any two points x, y on the new road system, will have direct distance no smaller that a constant factor times the distance through the new road system, i.e. for any x, y in the **new** road system, $d_{\hat{G}}(x, y) \leq C \cdot d_E(x, y)$. Here, $d_{\hat{G}}(x, y)$ is the distance between x and y along $\hat{G}$. The constant C, which appears above both in the size of the new road system and the comparison of distances is **independent** of the initial road system, and depends only on the dimension of the Euclidean space, d. (We conjecture that the same result holds in an infinite dimensional space.) To connect the above description to the statement of our theorem, take $K = G$, and $\hat{G} = \hat{\Gamma}$.

The proof is done via a stopping time argument on dyadic-like sets, which uses the Jones beta numbers and the Analyst's Traveling Salesman theorem. It is very geometric, and somewhat technical. Previous results of this type were only known for the plane, i.e. $d = 2$ (P. Jones, and later on, C. Bishop). There is related work in the computer science community (Kenyon-Kenyon, Das-Narasimhan), where only distances between points $x, y \in G$ are ever taken into account (as opposed to $x, y, \in \hat{G}$).

11. J. Azzam and R Schul, *Hard Sard: quantitative implicit function and extension theorems for Lipschitz maps*, Geom. Funct. Anal. **22** (2012), no. 5, 1062–1123.

 Suppose $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a Lipschitz function. Standard results tell us that near a.e. point $z \in f(\mathbb{R}^{n+m})$, the set $f^{-1}(z)$ is countably m-rectifiable. If f is sufficiently differentiable, we can do better: then Sard's theorem together with the implicit function theorem, tell us that for almost every image point z, we can foliate a neighborhood of a point in $f^{-1}(z)$ with m-dimensional fibers. Our goal was to give a quantitative version of this. Indeed, we show that for a Lipschitz $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, there is a large set $E \subset \mathbb{R}^{n+m}$ such that the m-dimensional fibers can be straightened out inside it. A bit more precisely: There is a (global) bi-Lipschitz function $g : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, such that if we write $F = f \circ g^{-1}$ and think of $\mathbb{R}^{n+m}$ as $\mathbb{R}^n \times \mathbb{R}^m$ then inside the set E, the function F is constant in the second variable and bi-Lipschitz in the first variable. See Figure 1. In fact, more is true. One can replace the image space $\mathbb{R}^n$ with **any** $n-$dimensional **metric space!** This result is new even for the case where the metric space is just $\mathbb{R}^n$ (because of the global, quantitative aspect of it). We should emphasize that the precise constants (size of E and bi-Lipschitz constants) depend on some notion of Hausdorff content of the image, and we omit the technicalities here. One should note that Kaufmann's 1979 construction of a function $f : [0,1]^3 \to [0,1]^2$ which is C^1, *surjective*, and of rank zero or one everywhere on the domain, which means that one has to be very careful here.

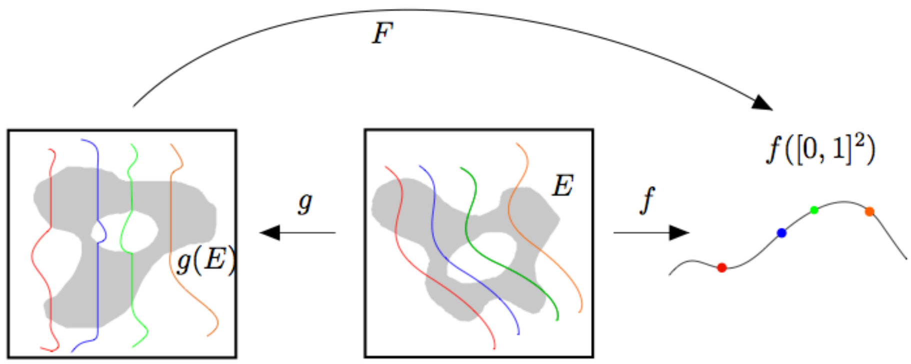

Figure 1: In the center square above, we have four rectifiable fibers that are preimages of four points in the image of f. We show that that there is a large region in the domain, E (denoted by the shaded area), so that the portions of these fibers that intersect E are sent to subsets of straight lines under g.

Another theorem is proven in this same paper, which is easier to precisely state. This is a **bi-Lipschitz extension theorem**, where the main point is that one does not have to increase the ambient dimension, at the expense of throwing out some of the domain. Let $0 < \kappa < 1$ be given. There is a constant $M = M(\kappa, d)$ such that if $f : \mathbb{R}^d \to \mathbb{R}^d$ is a 1-Lipschitz function, then the following hold.

(i) There are sets $E_1, ..., E_M$ such that

$$H^d_\infty(f([0,1]^d \setminus \bigcup E_i)) \lesssim_d \kappa$$

and for each E_i, $f|_{E_i}$ is $(l, 1)$-bi-Lipschitz with $l \sim \kappa$.

(ii) The sets E_i of part (i) may be chosen such that if $E_i \neq \emptyset$, there is $F_i : \mathbb{R}^d \to \mathbb{R}^d$ which is L-bi-Lipschitz, $L \sim_d \frac{1}{\kappa}$, so that

$$F_i|_{E_i} = f|_{E_i}.$$

Above, H^d_∞ is the usual d-dimensional Hausdorff content. What this means is that one can capture most of the image of a Lipschitz map with a finite collection of bi-Lipschitz images. The bi-Lipschitz maps are extensions of the original function restricted to various different pieces. (Compare with Theorem 5 in the 2009 Revista paper above.) The number of pieces and the bi-Lipschitz constants are controlled. Part (i) was in fact known by work of Peter

Jones (1988), but part (ii) is new and was used in our paper to answer some open question about uniform rectifiability. (In fact part (ii) was key in getting the function g from our implicit function theorem to be globally defined.)

This paper has so far been appealed to by two separate subfields in harmonic analysis. The first subfield studies harmonic measure on rough domains (See papers by Hofmann-Martell, Hofmann-Mitrea-Mitrea-Morris etc). The second subfield is the study of singular integrals group studies singular integrals along measurable vector fields (Gao). See the Google Scholar page for more details.

12. J. Azzam and R Schul, *A quantitative metric differentiation theorem*, Proc. Amer. Math. Soc. vol. 142 (2014), pages 1351-1357.

    It is easy to give a precise statement, but before I do that I should summarize and say that it is a quantitative version of metric differentiation, to complement the qualitative one by Kirchheim. Let $f : \mathbb{R}^d \to \mathcal{M}$ be a 1-Lipschitz function from Euclidean space to a general metric space. Let $Q \subset \mathbb{R}^n$ be a cube. Define

    $$\mathrm{md}(Q) := \frac{1}{\mathrm{side}(Q)} \inf_{\|\cdot\|} \sup_{x,y\in Q} \Big| \mathrm{dist}(f(x), f(y)) - \|x-y\| \Big|$$

    where the infimum is taken over all seminorms $\|\cdot\|$ on $\mathbb{R}^n$. The theorem we prove is that for any $\delta > 0$, and for each $R \in \Delta(\mathbb{R}^n)$,

    $$\sum \{\mathrm{volume}(Q) : Q \in \Delta(R), \mathrm{md}(3Q) > \delta\} \leq C_{\delta,n} \cdot \mathrm{volume}(R)\,.$$

    The constant $C_{\delta,n}$ does not depend on the metric space $\mathcal{M}$ or the function $f : \mathbb{R}^n \to \mathcal{M}$. ($\Delta(R)$ is the collection of all dyadic cubes inside R.)

13. M. Badger and R. Schul, *Multiscale analysis of 1-rectifiable measures: necessary conditions*, Math. Ann. 361 (2015), no. 3-4, 1055-1072.
    AND

14. M. Badger and R. Schul, *Two sufficient conditions for rectifiable measures*, Proc. Amer. Math. Soc. 144 (2016), no. 6, 2445-2454.
    AND

15. M. Badger and R. Schul, *Multiscale analysis of 1-rectifiable measures II: Characterizations*, Anal. Geom. Metr. Spaces 5 (2017), 1-39.

    In these three papers we investigate the notion of 1-rectifiability of a measure.

    Let ν be a positive and finite measure on $\mathbb{R}^n$. We say that ν is k-rectifiable if there is a countable collection of Lipschitz maps $f_i : [0,1]^k \to \mathbb{R}^n$ such that

    $$\nu(\mathbb{R}^n \setminus \cup f_i[0,1]^k)) = 0\,. \tag{11}$$

One can ask for geometric answers for *"When is ν k-rectifiable?"*. It is important to differentiate this from the question *"When is a set E k-rectifiable?"*, which corresponds to the case of $\nu = \mathcal{H}^k|_E$, or, more generally, the special case where $\nu \ll \mathcal{H}^k|_E$. The case $\nu \ll \mathcal{H}^k|_E$ is a very well studied, with work that started as early as the 1920's (Besicovitch) and had substantial contributions from many people, including Morse, Moore, Marstrand, Mattila and Preiss. Necessary and sufficient conditions came down to notions of *density* and *tangency* (appropriately defined). Since the 80's, a more quantitative approach has also been taken, and the question of *"When is a large potion of ν captured by a small number of 1-Lipschitz (or bi-Lipschitz) images?* Again, for the case where E is a set and $\nu \ll \mathcal{H}^k|_E$, this had contributions from many people, notably P. Jones, G. David and S. Semmes. A crucial observation is that in this case the maps f_i can always be of the form $x \mapsto A(x, g(x))$ where g is Lipschitz and A is affine.

In our 2010 paper(Garnet-Killip-Schul) an example was constructed of a probability measure ν on $\mathbb{R}^n$ $(n \geq 2)$ such that ν is $1-$rectifiable, and had some nice properties like being doubling (there is a $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, $\nu(B(x, 2r)) < C\nu(B(x, r))$). Recall that an implication of this is that $\nu(G) = 0$ whenever G is a Lipschitz graph, i.e. of the form $x \mapsto A(x, g(x))$ where g is Lipschitz and A is affine. Another property is that $\mathcal{H}^1|_\Gamma \perp \nu$ for any Lipschitz image Γ. While it is true that this measure is $1-$rectifiable, this follows from an explicit construction (done in that 2010 paper) of the functions f_i in eq. (11). All known methods for deciding whether this measure is rectifiable simply failed because of the above mutual singularity.

In these papers, M. Badger and myself gave a necessary and sufficient condition for a Borel measure on $\mathbb{R}^n$ to be 1-rectifiable. This confirmed a conjecture made by Peter Jones. We define for every locally finite Borel measure ν on $\mathbb{R}^n$ and a function $\tilde{J}(\nu, x)$, which is a variant of the Jones function. It is a weighted sum over all scales of how well ν is approximated by a linear regression model on each scale. We show that the rectifiable and unrectifiable parts can be discerned by looking at the finiteness of $\{\tilde{J}(\nu, x)$.

Two note are worth making: one is that we leave the question of $2, 3, 4$...-rectifiability of a measure wide open, and the other is that part of resolving the 1-rectifiability question involved proving a variant of Peter Jones' Traveling Salesman Theorem.

16. S. Li and R. Schul, *The traveling salesman problem in the Heisenberg group: upper bounding curvature*, Trans. Amer. Math. Soc. 368 (2016), no. 7, 4585-4620.

    AND

17. S. Li and R. Schul, *An upper bound for the length of a Traveling Salesman path in the Heisenberg group.* , Rev. Mat. Iberoam. 32 (2016), no. 2, 391-417.

    Let $\mathbb{H}$ denote the Heisenberg group, endowed with the Carnot-Carathéodory distance and $E \subseteq \mathbb{H}$ be any subset. Then we denote

$$\beta_{E,\mathbb{H}}(B) = \inf_L \sup_{x \in E \cap B} \frac{\operatorname{dist}(x, L)}{\operatorname{diam}(B)},$$

with the infimum is taken over all *horizontal lines* L (to be defined in the next section). In these papers we show the following.

Theorem 9. *There is a constant $C > 0$ such that for any rectifiable curve Γ the following holds. We have*

$$\int_{\mathbb{H}} \int_0^{+\infty} \beta_{\Gamma,\mathbb{H}}(B(x,t))^4 \frac{dt}{t^4} d\mathcal{H}^4(x) \leq C\mathcal{H}^1(\Gamma). \tag{12}$$

In the other direction, we prove the following theorem.

Theorem 10. *Let $p < 4$ be fixed. There is a constant $C = C(p) > 0$ such that for any set $E \subseteq \mathbb{H}$ if*

$$\mathrm{diam}(E) + \int_{\mathbb{H}} \int_0^{+\infty} \beta_{E,\mathbb{H}}(B(x,t))^p \frac{dt}{t^4} d\mathcal{H}^4(x) < \infty,$$

then there exists a rectifiable curve $\Gamma \supset E$ such that

$$\mathcal{H}^1(\Gamma) \leq C\left(\mathrm{diam}(E) + \int_{\mathbb{H}} \int_0^{+\infty} \beta_{E,\mathbb{H}}(B(x,t))^p \frac{dt}{t^4} d\mathcal{H}^4(x)\right). \tag{13}$$

We do not know if Theorem 10 is still true when $p = 4$.

In $\mathbb{R}^d$, the analogous theorems hold with the same power $r = 2$ for both Theorem 10 and 9 (t^4 is replaced by t^n as n is the Hausdorff dimension of $\mathbb{R}^n$; affine lines replace horizontal lines). These are shown in the papers by Jones (1990) and Okikiolu (1992) mentioned just before Theorem 1 at the start of this bibliographic document. If one discretizes the integral to a sum in an appropriate way, the same holds for an infinite dimensional Hilbert space (see Theorems 1 and 2). In a general metric space, with no assumptions on the set E, there is an analogue of Theorem 10 by Hahlomaa (2005), however the analogue of Theorem 9 is false in that setting (Schul, 2007). If one adds the assumption that E is 1-Ahlfors-regular (i.e. it supports a measure with linear upper and lower bounds on its growth) then, in this general metric setting analogues of both Theorem 10 and 9 hold (a combination of works of Schul and Hahlomaa). Finally we note that extensive work on the Euclidean case had been done by many people in the setup where one approximates with k-planes rather than lines. and that there is a very deep connection between these geometric questions and singular integral operators. See for example works of David, Semmes, Tolsa and others.

18. G. C. David and R. Schul, *The analyst's traveling salesman theorem in graph inverse limits*, Ann. Acad. Sci. Fenn. Math. 42 (2017), 649-692.

We prove a version of Peter Jones' Analyst's traveling salesman theorem in a class of highly non-Euclidean metric spaces introduced by Laakso and generalized by Cheeger-Kleiner. These spaces are constructed as inverse limits of metric graphs, and include examples which are doubling and have a Poincare inequality. We show that a set in one of these spaces is contained in a rectifiable curve if and only if it is quantitatively "flat" at most locations and scales, where flatness is measured with respect to so-called monotone geodesics. This provides a first examination of quantitative rectifiability within these spaces.

19. J. Azzam ad R. Schul, *An analyst's traveling salesman theorem for sets of dimension larger than one*, Accepted to Mathematische Annalen. ArXiv preprint arXiv:1609.02892 .

Jonas Azzam and the author have recently shown a version of Jones' Traveling Salesman Theorem (TST) for sets of dimension larger than 1. Much work on this exists already by G. David and S. Semmes (and many others), but they assume Ahlfors regularity of the underlying set (i.e., that the $\mathcal{H}^k$ measure on it scales like r^k when measuring balls of radius r around a point in the set). We will state our Theorems with Azzam, but first we need some definitions. We will then discuss new problems stemming from our work.

Definition 11. *Given two closed sets E and F, and a cube or ball Q, we denote*

$$d_Q(E,F) = \frac{2}{\mathrm{diam} Q} \max\left\{ \sup_{y\in E\cap Q} \mathrm{dist}(y,F), \sup_{y\in F\cap Q} \mathrm{dist}(y,E) \right\}, \text{ and}$$

$$\vartheta_E^d(Q) = \inf\{d_Q(E,L) : L \text{ is a } d\text{-dimensional plane in } \mathbb{R}^n\}.$$

We say E is (ϵ,d)-Reifenberg flat (or just ϵ-Reifenberg flat when the dimension is given) if $\vartheta_E^d(3Q) < \epsilon$ for all $Q \in \Delta(\mathbb{R}^n)$ with $Q \cap E \neq \emptyset$.

Definition 12. *Let $E \subseteq \mathbb{R}^n$ and $\epsilon, A > 0$. We define define*

$$\Theta_E^{d,\Delta,\epsilon} := \sum\{\mathrm{diam}(Q)^d : Q \in \Delta,\ Q\cap E\cap B(0,1) \neq \emptyset \text{ and } \vartheta_E(3Q) \geq \epsilon\}.$$

Above, Δ is the collection of all dyadic cubes of side at most 1.

Remark 13. *If E is an ϵ-Reifenberg flat then $\Theta_E^{d,\Delta,\epsilon} = 0$.*

Definition 14. *A set $E \subseteq \mathbb{R}^n$ is said to be (c,d)-lower content regular in a ball B if*

$$\mathcal{H}^d_\infty(E\cap B(x,r)) \geq cr^d \text{ for all } x \in E\cap B \text{ and } r \in (0, r_B).$$

We now define our β-number that, firstly, is in some sense an L^p-average of distances to a plane (rather than an L^∞-norm), and secondly, doesn't rely on the underlying measure on the set in question. This seems somewhat self-contradictory, but we are able to achieve this by manipulating the definition of $\beta^d_{\mathcal{H}^d|_E,p}$ and "integrate" with respect to $\mathcal{H}^d_\infty$ (rather than $\mathcal{H}^d$) using a Choquet integral.

Definition 15. *For arbitrary set E, and Q a cube,*

$$\beta_E^{d,p}(Q) = \inf_L \left(\frac{1}{\operatorname{diam}(Q)^d} \int_0^1 \mathcal{H}_\infty^d(\{x \in Q \cap E : \operatorname{dist}(x, L) > t \cdot \operatorname{diam}(Q)\}) t^{p-1} dt \right)^{\frac{1}{p}}$$

where L goes over all d-planes.

Note that if we assume E is Ahlfors d-regular, then this quantity is comparable to the David-Semmes type quantity, $\beta^{d,p}_{\mathcal{H}^d|_E, p}$. The main result of Azzam and the author in this paper is the following.

Theorem 16. *Let $1 \le d < n$, $C_0 > 1$. Let $1 \le p < p(d)$. Let $E \subseteq \mathbb{R}^n$ be a closed set containing 0. Suppose that E is (c, d)-lower content regular in $B(0, 1)$. There is $\epsilon_0 = \epsilon_0(n, p, c) > 0$ such that for $0 < \epsilon < \epsilon_0$. Then*

$$1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^{d,p}(C_0 Q)^2 \operatorname{diam}(Q)^d \lesssim_{C_0,n,\epsilon,p,c} \mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d,\Delta,\epsilon}. \tag{14}$$

The appearance of $\Theta_E^{d,\Delta,\epsilon}$ is quite natural in the following sense. One may estimate the length of the shortest curve containing a set E by Jones' TST. This length, however, could be much bigger than the length of E. The set E can be seen as the curve Γ punctured by holes, which $\Theta_E^{d,\Delta,\epsilon}$ accounts for. This happens since ϑ_E^d also measures how far an optimal plane is from E, and hence, if ϑ_E^d is large yet $\beta_E^{d,p}$ is small, this means that E is very flat but contains a large d-dimensional hole. Azzam and the author also show a converse to Theorem 16 as follows.

Theorem 17. *Let $1 \le d < n$. Let $1 \le p \le \infty$ and $E \subseteq \mathbb{R}^n$ be (c, d)-lower content regular in $B(0, 1)$ such that $0 \in E$. Let $\epsilon > 0$ be given. Then for C_0 sufficiently large (depending only on n),*

$$\mathcal{H}^d(E \cap B(0, 1)) + \Theta_E^{d,\Delta,\epsilon} \lesssim_{n,c,C_0,\epsilon} 1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^{d,p}(C_0 Q)^2 \operatorname{diam}(Q)^d. \tag{15}$$

Furthermore, if the right hand side of (15) *is finite, then E is d-rectifiable*

Remark 18. *The proof of Theorem 17 contains more information than presented in its statement. The rectifiability of E comes about from the construction of a sequence of surfaces, which are bi-Lipschitz images of d-dimensional cubes. The bi-Lipschitz constant and sizes of cubes are controlled. For example, one may slightly modify the construction to yield a connected, rectifiable set Γ such that $\mathcal{H}^d(E \setminus \Gamma) = 0$ and for all $x \in \Gamma$, $r \le 1$ we have $\mathcal{H}^d_\infty(\Gamma \cap B(x, r)) \ge c' r^d$ for some (explicit) $c' > 0$. This Γ will have*

$$\mathcal{H}^d(\Gamma) \le C(n, c) \left(1 + \sum_{\substack{Q \in \Delta \\ Q \cap E \cap B(0,1) \neq \emptyset}} \beta_E^{d,p}(C_0 Q)^2 \operatorname{diam}(Q)^d \right).$$

In fact, one obtains a coronization similar to that in David-Semmes' body of work, but with bi-Lipschitz surfaces in place of Lipschitz graphs.

Finally, we note that the $\Theta_E^{d,\Delta,\epsilon}$ quantity is subsumed by the $\mathcal{H}^d$ quantity in some natural situations.

Definition 19 (Condition B)**.** *We will say $E \subseteq \mathbb{R}^n$ satisfies Condition B for some $c > 0$ if for all $x \in E$ and $r > 0$, one can find two balls of radius rc contained in $B(x,r)$ in two different components of E^c.*

Usually, this definition also assumes E is Ahlfors regular, and then there are other proofs that, in this situation, E is uniformly rectifiable. If E has locally finite $\mathcal{H}^{n-1}$-measure, then one can show that, for any ball B centered on E, there is a Lipschitz graph Γ so that $\mathcal{H}^{n-1}(\Gamma \cap B \cap E) \geq r_B^{n-1}$ with Lipschitz constant depending on $\mathcal{H}^{n-1}(B \cap E)$.

Theorem 20 (David-Semmes)**.** *Let $E \subseteq \mathbb{R}^n$ satisfy Condition B for some $c > 0$. Then for all $x \in E$, and $\epsilon, r > 0$, we have $\Theta_E^{n-1,\Delta,\epsilon}(x,r) \leq C(\epsilon,n)\mathcal{H}^{n-1}(E \cap B(x,r))$.*

Remark 21. *In our paper we present examples which show why other notions of β numbers are not good enough to get both of the main theorems, and why Θ is needed.*

A fascinating and hard question is summed up in the following problem. *Under what conditions can the conclusion of rectifiability in Theorem 17 be improved to: "E can be fully covered by a single Lipschitz image of $[0,1]^k$?" What if we replace Lipschitz by a weaker notion?*