The geometry of Radon-Nikodym Lipschitz differentiability spaces, II

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RNP Lipschitz differentiability spaces

Let (X, d) be a metric space, V be a Banach space, and $\varphi : X \to \mathbb{R}^n$ (a chart). We say $f : X \to V$ is φ-differentiable at $x_0 \in X$ if there exists a *unique* $Df(x_0) \in L(\mathbb{R}^n, V)$ so that

$$f(x) = f(x_0) + Df(x_0)(\varphi(x) - \varphi(x_0)) + o(d(x, x_0)).$$

Definition

A metric measure space (X, d, μ) is a Radon-Nikodym Lipschitz differentiability space (RNP-LDS) if there is a Lipschitz chart $\varphi : X \to \mathbb{R}^n$ so that for every Banach space V with the Radon-Nikodym property and every Lipschitz function $f : X \to V$ is φ-differentiable at μ-a.e. $x \in X$.

Upper gradients

Let (X, d, μ) be a (rectifiably) path connected space. A measurable function $\rho : X \to [0, \infty]$ is an upper gradient for a Lipschitz function $f : X \to \mathbb{R}$ if for every rectifiable curve $\gamma : [a, b] \to X$ we have

$$|f(\gamma(b)) - f(\gamma(a))| \leq \int_\gamma \rho \, ds.$$

Upper gradients are not unique, but they always exist for path connected spaces by considering $\rho \equiv \infty$. If X is geodesic, then we may take $\rho \equiv L$ when f is L-Lipschitz.

Poincaré inequality

Definition (Heinonen-Koskela)

A path connected space (X, d, μ) satisfies a Poincaré inequality if there exists $C \geq 1$ and $p \in [1, \infty)$ so that for every Lipschitz function $f : X \to \mathbb{R}$ with upper gradient $\rho : X \to [0, \infty]$, we have

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,Cr)} \rho^p \, d\mu \right)^{1/p}, \quad \forall x \in X, r > 0.$$

"Path fluctuations control metric fluctuations"

A PI space is metric measure space that satisfies a Poincaré inequality and is doubling, *i.e.* there exists $C \geq 1$ so that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)), \qquad \forall x \in X, r > 0.$$

PI and differentiability

Theorem (Cheeger-Kleiner)

PI spaces are Radon-Nikodym Lipschitz differentiability spaces.

Q: Does the converse hold? A: No.

PI spaces are path connected. Positive measure subsets of RNP-LDS are RNP-LDS (with induced measure and metric). Thus, fat Cantor sets of $[0, 1]$ are RNP-LDS, but totally disconnected.

Need to relax doubling and PI.

Need to relax line integral and upper gradient.

Line integrals

Let (X, d, μ) be a metric measure space (possibly disconnected) and $f : X \to \mathbb{R}$ be 1-Lipschitz and $\rho : X \to [0, \infty]$ be measurable.

Let $\gamma : K \to X$ be a 1-Lipschitz map so that $K \subset \mathbb{R}$ is compact (a "fragment") with $a = \min K$ and $b = \max K$. We want something like

$$|f(\gamma(b)) - f(\gamma(a))| \leq \int_\gamma \rho \, ds.$$

We can still make sense of

$$\int_\gamma \rho \, ds := \int_K \rho(\gamma(s))|\gamma'(s)| \, ds.$$

However, f can fluctuate across gaps of X and thus K.

Example: Let $X \subset [0, 1]$ be a fat Cantor set and $f(t) = \int_0^t \chi_{[0,1]\setminus X} \, dx$.

Line integrals (cont.)

Let (c, d) be a gap in K. As f and γ are 1-Lipschitz, we have

$$|f(\gamma(c)) - f(\gamma(d))| \leq |c - d|.$$

Define the $*$-integral

$$\int_\gamma^* \rho := \int_K \rho \; ds + |[a, b] \backslash K|.$$

We then have that

$$|f(\gamma(b)) - f(\gamma(a))| \leq \int_\gamma^* 1.$$

We say $\rho : X \to [0, 1]$ is a $*$-upper gradient of f if for all fragments γ

$$|f(\gamma(b)) - f(\gamma(a))| \leq \int_\gamma^* \rho.$$

Asymptotic nonhomogeneous Poincaré inequality

A metric measure space (X, d, μ) satisfies an asymptotic nonhomogeneous Poincaré inequality if there exist $C \geq 1$ and continuous increasing moduli $\zeta_x, o_x : [0, \infty) \to [0, \infty)$ for μ-a.e. $x \in X$ so that for every 1-Lipschitz $f : X \to \mathbb{R}$ with $*$-u.g. $\rho : X \to [0, 1]$, we have

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq r\zeta_x \left(\fint_{B(x,Cr)} \rho \, d\mu \right) + o_x(r).$$

Here, ζ_x and o_x satisfy

$$\lim_{t \to 0} \frac{o_x(t)}{t} = 0, \qquad \lim_{t \to 0} \zeta_x(t) = 0.$$

Characterization of RNP-LDS

Definition

A metric measure space (X, d, μ) is pointwise doubling if for μ-a.e. $x \in X$,

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty.$$

Theorem (Bate-L.)

A metric measure space is a Radon-Nikodym Lipschitz differentiability space if and only if it is pointwise doubling and satisfies an asymptotic nonhomogeneous Poincaré inequality.