The Structure of the Singular Set of a Two-Phase Free Boundary Problem for Harmonic Measure

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Singular Sets and Regular Sets

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Plethora of other examples: zero sets of solutions to elliptic PDE (Cheeger-Naber-Valtorta '15), support of uniform measures (Nimer '15), solutions to the thin obstacle problem (Garofalo-Petrosyan '09).

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Use this to understand two-phase problem for harmonic measure.

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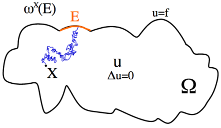

Harmonic measure at X.

Credit Wikipedia

Given
$f \in C(\partial\Omega)$. $\exists U_f \in C^2(\Omega) \cap C(\overline{\Omega})$ which satisfies:

$$\Delta U_f(x) = 0, \ x \in \Omega$$
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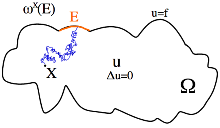

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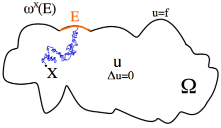

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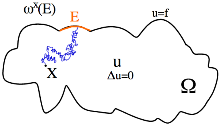

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Harnack inequality $\Rightarrow \omega^X << \omega^Y << \omega^X$. Will omit dependence on pole.

Two-phase Free Boundary Problems

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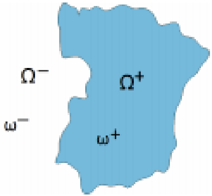

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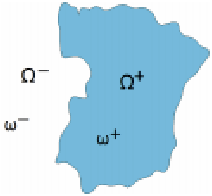

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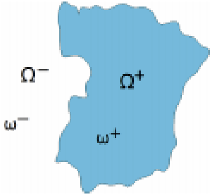

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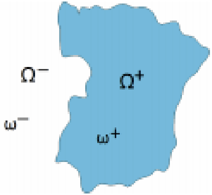

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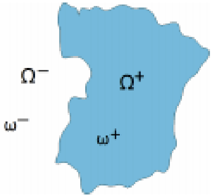

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$\Omega \subset \mathbb{R}^2$, use complex analysis (Garnett and Marshall (chpt 6)).

What are NTA domains?

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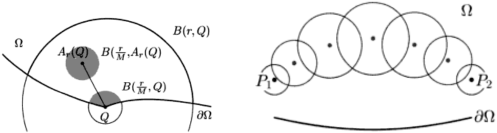

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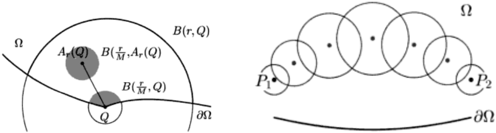

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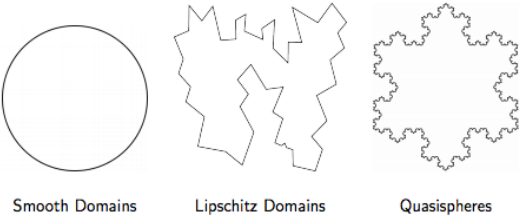

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DEFINITION ((PSEUDO)-BLOWUPS)

A set, C, is a **pseudo-blowup** of Γ if there exists $Q_i \in \Gamma, r_i \downarrow 0$ such that

$$\frac{\Gamma - Q_i}{r_i} \equiv \Gamma_i \to C.$$

If $Q_i \equiv Q$, call it a **blowup**.

FIGURE: Blowing up at a point. Picture courtesy of Matthew Badger.

Blowup Analysis

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IMPORTANT: May be multiple blowups at a point (for different $\{r_i\}$).

Blowups of the Two-Phase Problem

Theorem (Kenig-Toro '06)

Let $\Omega^{\pm} \subset \mathbb{R}^n$ *be complementary NTA with* $\log(h) \in \mathrm{VMO}(d\omega^+)$ *(almost continuous) then every pseudo-blowup of* Γ *is actually the zero set of a degree* $\leq d_0$ *harmonic polynomial,* p*.*

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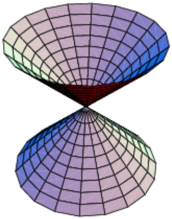

$h(X) = x_1^2 + x_2^2 - x_3^2 - x_4^2$ is a harmonic polynomial s.t. $\{h > 0\}$ and $\{h < 0\}$ are NTA. Credit: Mathematica

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Let $\Omega^{\pm} \subset \mathbb{R}^n$ *be complementary NTA domains and assume* $\log(\frac{d\omega^-}{d\omega^+}) \in \mathrm{VMO}(d\omega^+)$. $\exists d_0 \in \mathbb{N}$ *s.t.* $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_{d_0}$, *where:*

- *If* $Q \in \Gamma_k$, *then any blowup of* Γ *at* Q *is the zero set of a degree* k *homogenous harmonic polynomial (not necessarily unique!).*
- $\overline{dim}_M \Gamma \backslash \Gamma_1 \leq n-3$. $\Gamma \backslash \Gamma_1$ *is the* **singular set**.
- *For any* $k \leq d_0$: $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$. *is open inside of* Γ
- *For any* $k \leq d_0/2$: $\dim_H \Gamma_2 \cup \Gamma_4 \cup \ldots \cup \Gamma_{2k} \leq n-4$.

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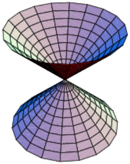

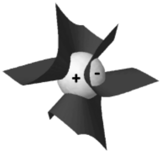

These two examples $x_1^2 + x_2^2 - x_3^2 - x_4^2$ and $x_1^2(x_2 - x_3) + x_2^2(x_3 - x_1) + x_3^2(x_1 - x_2) + x_1x_2x_3$ show that the above dimension bounds are sharp. Credit: Mathematica and M. Badger

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- **Detectability:** Let $k \leq \ell$ and $C, \delta > 0$ be uniform constants. If p, h are harmonic polynomials of degree k, ℓ, respectively, $\{p = 0\} \cap B(x, r)$ is within δr of $\{h = 0\} \cap B(x, r)$, then for every $s \in (0, 1)$ there is a degree k polynomial, p_s, such that $\{p_s = 0\} \cap B(x, rs)$ is within $Crs^{1+1/k}$ of $\{h = 0\} \cap B(x, rs)$.

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  - If you are close to a degree k polynomial at one scale, you get closer at smaller scales. ("improvement of flatness"-type result)
- **Dimension Estimates:** for every $\delta > 0$, $\exists C > 0$ such that for all harmonic polynomials, p, of degree $\leq d_0$,

$$\mathrm{Vol}(\{x \in B(0, 1/2) \mid p(x) = 0, \mathrm{dist}(x, \mathcal{S}(p)) < r\}) \leq Cr^{3-\delta},$$

where $\mathcal{S}(p) = \{x_0 \mid p(x_0) = 0 = Dp(x_0)\}$, is the singular set of p.

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- Compactness argument shows that if $\{p = 0\}$ is very close to **some** degree k harmonic polynomial, then it must be near the first k terms of its Taylor series.

Theorem (Cheeger-Naber-Valtorta '15)

If $u : B(0,1) \to \mathbb{R}$ *is a harmonic function with* $u(0) = 0$ *and* $\frac{\int_{B(0,1)} |\nabla u|^2 dx}{\int_{\partial B_1} u^2 \, d\sigma} \leq \Lambda$*, then for every* $\eta > 0$ *and* $k \leq n - 2$*,*

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- "Regular points" depend on only one direction at infinitesimal scales. So if $k < n-1$ we are looking at singular points.
- **We show:** Singular points $\mathcal{S}(p) \subset \mathcal{S}^{n-3}_{\eta,r}$ for all r and η small enough.

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Theorem (Cheeger-Naber-Valtorta '15)

If $u : B(0,1) \to \mathbb{R}$ is a harmonic function with $u(0) = 0$ and $\frac{\int_{B(0,1)} |\nabla u|^2 dx}{\int_{\partial B_1} u^2 \; d\sigma} \leq \Lambda$, then for every $\eta > 0$ and $k \leq n-2$,

$$\mathrm{Vol}(\{x \in B(0,1/2) \mid \mathrm{dist}(x, \mathcal{S}^k_{\eta,r}(u))\}) \leq C(n,\Lambda,\eta) r^{n-k-\eta}.$$

- $\mathcal{S}^k_{\eta,r}(u)$ are the points at which u "depends" on more than $n-k$ variables at small scales (i.e. has k or more translational symmetries).
- "Regular points" depend on only one direction at infinitesimal scales. So if $k < n-1$ we are looking at singular points.
- **We show:** Singular points $\mathcal{S}(p) \subset \mathcal{S}^{n-3}_{\eta,r}$ for all r and η small enough.
- **Proof:** Blow-up argument and no homogenous harmonic polynomial splits $\mathbb{R}^2$ into two connected components.

Some Open Questions/Future Work

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3. $\log(\frac{d\omega^-}{d\omega^+}) \in \mathrm{VMO}$
    - Unique blowups at points?
    - Is Γ_k closed?
    - We need to understand better the zero sets of harmonic polynomials **which split space into two NTA components**.

Thank You For Listening!