Removability, Rigidity of Circle Domains and Koebe's Conjecture

Malik Younsi, Stony Brook University

2016 AMS Spring Meeting

Circle domains

Circle domains

Definition

Circle domains

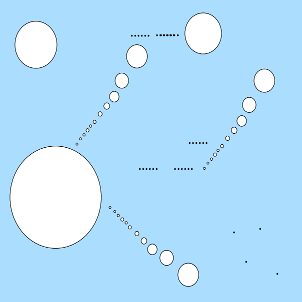
Definition

A domain Ω in the Riemann sphere $\widehat{\mathbb{C}}$ is a **circle domain** *if every connected component of its boundary is either a round circle or a point.*

Circle domains

Definition

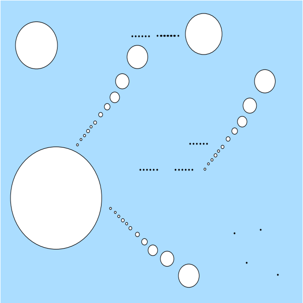
A domain Ω in the Riemann sphere $\widehat{\mathbb{C}}$ is a **circle domain** *if every connected component of its boundary is either a round circle or a point.*



Circle domains

Definition

A domain Ω in the Riemann sphere $\widehat{\mathbb{C}}$ is a **circle domain** *if every connected component of its boundary is either a round circle or a point.*



- The boundary of any circle domain contains at most countably many circles.

The Koebe Uniformization Conjecture

The Koebe Uniformization Conjecture

Conjecture (Koebe, 1909)

Every domain $D \subset \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.

The Koebe Uniformization Conjecture

Conjecture (Koebe, 1909)

Every domain $D \subset \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.

The conjecture is true if D

The Koebe Uniformization Conjecture

Conjecture (Koebe, 1909)

Every domain $D \subset \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.

The conjecture is true if D

- has finitely many boundary components (Koebe, 1918)

The Koebe Uniformization Conjecture

Conjecture (Koebe, 1909)

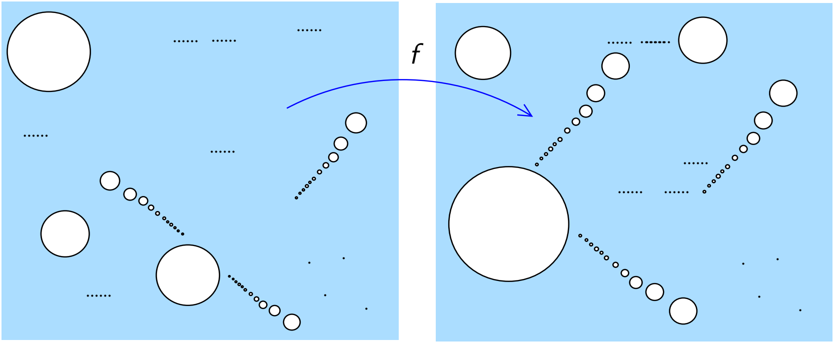
Every domain $D \subset \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.

The conjecture is true if D

- has finitely many boundary components (Koebe, 1918)
- has countably many boundary components (He–Schramm, 1993).

Uniqueness of the Koebe map

Uniqueness of the Koebe map



Conformal rigidity

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ is **conformally rigid** *if every conformal map of Ω onto another circle domain is the restriction of a Möbius transformation.*

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ *is* **conformally rigid** *if every conformal map of* Ω *onto another circle domain is the restriction of a Möbius transformation.*

Ω is conformally rigid if

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ is **conformally rigid** *if every conformal map of Ω onto another circle domain is the restriction of a Möbius transformation.*

Ω is conformally rigid if

- it has finitely many boundary components (Koebe, 1918)

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ *is* **conformally rigid** *if every conformal map of* Ω *onto another circle domain is the restriction of a Möbius transformation.*

Ω is conformally rigid if

- it has finitely many boundary components (Koebe, 1918)
- it has countably many boundary components (He–Schramm, 1993)

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ is **conformally rigid** *if every conformal map of Ω onto another circle domain is the restriction of a Möbius transformation.*

Ω is conformally rigid if

- it has finitely many boundary components (Koebe, 1918)
- it has countably many boundary components (He–Schramm, 1993)
- it has σ-finite length boundary (He–Schramm, 1994).

Conformally removable sets

Conformally removable sets

Definition

Conformally removable sets

Definition

Let $E \subset \mathbb{C}$ be compact.

Conformally removable sets

Definition

Let $E \subset \mathbb{C}$ be compact.
We say that E is **conformally removable** *if every homeomorphism of $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

Conformally removable sets

Definition

Let $E \subset \mathbb{C}$ be compact.
We say that E is **conformally removable** *if every homeomorphism of $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

- Removable : sets of σ-finite length, quasicircles, Hölder curves

Conformally removable sets

Definition

Let $E \subset \mathbb{C}$ be compact.
We say that E is **conformally removable** *if every homeomorphism of $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

- Removable : sets of σ-finite length, quasicircles, Hölder curves
- Non-removable : sets of positive area

Conformally removable sets

Definition

Let $E \subset \mathbb{C}$ be compact.
We say that E is **conformally removable** *if every homeomorphism of $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

- Removable : sets of σ-finite length, quasicircles, Hölder curves
- Non-removable : sets of positive area
- There exist removable sets of Hausdorff dimension two and non-removable sets of Hausdorff dimension one

Conformally removable sets

Definition

Let $E \subset \mathbb{C}$ be compact.
We say that E is **conformally removable** *if every homeomorphism of $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

- Removable : sets of σ-finite length, quasicircles, Hölder curves
- Non-removable : sets of positive area
- There exist removable sets of Hausdorff dimension two and non-removable sets of Hausdorff dimension one
- The complement of any non-removable Cantor set is a non-rigid circle domain.

Rigidity conjecture

Rigidity conjecture

Conjecture (He–Schramm 1994)

Rigidity conjecture

Conjecture (He–Schramm 1994)

Let Ω be a circle domain.

Rigidity conjecture

Conjecture (He–Schramm 1994)

Let Ω be a circle domain. The following are equivalent :

Rigidity conjecture

Conjecture (He–Schramm 1994)

Let Ω be a circle domain. The following are equivalent :

(A) Ω *is conformally rigid;*

Rigidity conjecture

Conjecture (He–Schramm 1994)

Let Ω be a circle domain. The following are equivalent :

(A) *Ω is conformally rigid;*

(B) *The boundary of Ω is conformally removable;*

Rigidity conjecture

Conjecture (He–Schramm 1994)

Let Ω be a circle domain. The following are equivalent :

(A) *Ω is conformally rigid;*

(B) *The boundary of Ω is conformally removable;*

(C) *Every Cantor set contained in the boundary of Ω is conformally removable.*

Rigidity conjecture

Conjecture (He–Schramm 1994)

Let Ω be a circle domain. The following are equivalent :

(A) *Ω is conformally rigid;*

(B) *The boundary of Ω is conformally removable;*

(C) *Every Cantor set contained in the boundary of Ω is conformally removable.*

Remarks :

Rigidity conjecture

Conjecture (He–Schramm 1994)

Let Ω be a circle domain. The following are equivalent :

(A) *Ω is conformally rigid;*

(B) *The boundary of Ω is conformally removable;*

(C) *Every Cantor set contained in the boundary of Ω is conformally removable.*

Remarks :

- **(B)** $\Rightarrow$ **(C)** is trivial

Rigidity conjecture

Conjecture (He–Schramm 1994)

Let Ω be a circle domain. The following are equivalent :

(A) *Ω is conformally rigid;*

(B) *The boundary of Ω is conformally removable;*

(C) *Every Cantor set contained in the boundary of Ω is conformally removable.*

Remarks :

- **(B)** $\Rightarrow$ **(C)** is trivial
- If there are no circles in $\partial\Omega$, then **(A)** $\Rightarrow$ **(B)**.

First Main Theorem

First Main Theorem

Theorem (Y. (2015))

First Main Theorem

Theorem (Y. (2015))

Let Ω be a circle domain whose boundary is the union of countably many circles and countably many totally disconnected compact sets.

First Main Theorem

Theorem (Y. (2015))

Let Ω be a circle domain whose boundary is the union of countably many circles and countably many totally disconnected compact sets.
Then **(B)** *and* **(C)** *are equivalent.*

First Main Theorem

Theorem (Y. (2015))

Let Ω be a circle domain whose boundary is the union of countably many circles and countably many totally disconnected compact sets.
Then **(B)** *and* **(C)** *are equivalent.*

- The assumption holds if boundary circles don't accumulate *too much* on point boundary components.

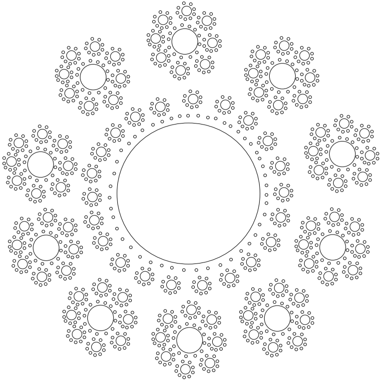
First Main Theorem

Theorem (Y. (2015))

Let Ω be a circle domain whose boundary is the union of countably many circles and countably many totally disconnected compact sets.
Then **(B)** *and* **(C)** *are equivalent.*

- The assumption holds if boundary circles don't accumulate *too much* on point boundary components.
- The assumption does not hold if boundary circles accumulate everywhere.

A Sierpinski-type circle domain



Countable unions of certain conformally removable sets

Countable unions of certain conformally removable sets

The theorem is a consequence of the following more general result.

Countable unions of certain conformally removable sets

The theorem is a consequence of the following more general result.

Theorem (Y. (2015))

Countable unions of certain conformally removable sets

The theorem is a consequence of the following more general result.

Theorem (Y. (2015))

Let E be a compact plane set of the form

$$E = \bigcup_{j=1}^{\infty} \Gamma_j \cup \bigcup_{k=1}^{\infty} E_k,$$

where each Γ_j is a quasicircle and each E_k is a totally disconnected compact set.

Countable unions of certain conformally removable sets

The theorem is a consequence of the following more general result.

Theorem (Y. (2015))

Let E be a compact plane set of the form

$$E = \bigcup_{j=1}^{\infty} \Gamma_j \cup \bigcup_{k=1}^{\infty} E_k,$$

where each Γ_j is a quasicircle and each E_k is a totally disconnected compact set.
Then E is conformally removable if and only if every E_k is conformally removable.

Countable unions of certain conformally removable sets

The theorem is a consequence of the following more general result.

Theorem (Y. (2015))

Let E be a compact plane set of the form

$$E = \bigcup_{j=1}^{\infty} \Gamma_j \cup \bigcup_{k=1}^{\infty} E_k,$$

where each Γ_j is a quasicircle and each E_k is a totally disconnected compact set.
Then E is conformally removable if and only if every E_k is conformally removable.

- Still open whether the union of two conformally removable sets is conformally removable (Jones–Smirnov, 2000).

Second Main Theorem

Second Main Theorem

It is not difficult to prove that

Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable

Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

Theorem (Y. (2015))

A circle domain Ω is conformally rigid if and only if it is quasiconformally rigid.

Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

Theorem (Y. (2015))

A circle domain Ω is conformally rigid if and only if it is quasiconformally rigid.

Corollary

Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

Theorem (Y. (2015))

A circle domain Ω is conformally rigid if and only if it is quasiconformally rigid.

Corollary

Let Ω be a circle domain and let f be a quasiconformal mapping of the sphere which maps Ω onto another circle domain $f(\Omega)$.

Second Main Theorem

It is not difficult to prove that

- a compact set is conformally removable if and only if it is quasiconformally removable
- quasiconformal mappings of the plane preserve conformally removable sets.

Theorem (Y. (2015))

A circle domain Ω is conformally rigid if and only if it is quasiconformally rigid.

Corollary

Let Ω be a circle domain and let f be a quasiconformal mapping of the sphere which maps Ω onto another circle domain $f(\Omega)$. If Ω is conformally rigid, then $f(\Omega)$ is also conformally rigid.

An important lemma

An important lemma

Lemma

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on Ω, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on Ω, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - μ satisfies the (strong) David condition :
    $m(\{z \in \widehat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\frac{\alpha}{\epsilon}}} \qquad (\epsilon > 0).$

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on Ω, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - μ satisfies the (strong) David condition :
    $m(\{z \in \widehat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\frac{\alpha}{\epsilon}}} \qquad (\epsilon > 0).$
  - μ is **invariant** with respect to the Schottky group of Ω.

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on Ω, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - μ satisfies the (strong) David condition :
    $m(\{z \in \widehat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\frac{\alpha}{\epsilon}}} \qquad (\epsilon > 0).$
  - μ is **invariant** with respect to the Schottky group of Ω.
- Use David's theorem to obtain a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_{\overline{z}} f / \partial_z f = \mu$ a.e.

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on Ω, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - μ satisfies the (strong) David condition :
    $m(\{z \in \widehat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\frac{\alpha}{\epsilon}}} \qquad (\epsilon > 0).$
  - μ is **invariant** with respect to the Schottky group of Ω.
- Use David's theorem to obtain a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_{\overline{z}} f / \partial_z f = \mu$ a.e.
- Deduce that

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on Ω, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - μ satisfies the (strong) David condition :
    $m(\{z \in \widehat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\frac{\alpha}{\epsilon}}} \qquad (\epsilon > 0).$
  - μ is **invariant** with respect to the Schottky group of Ω.
- Use David's theorem to obtain a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_{\overline{z}} f / \partial_z f = \mu$ a.e.
- Deduce that
  - f is conformal on Ω

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on Ω, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - μ satisfies the (strong) David condition :
    $m(\{z \in \widehat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\frac{\alpha}{\epsilon}}} \qquad (\epsilon > 0).$
  - μ is **invariant** with respect to the Schottky group of Ω.
- Use David's theorem to obtain a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_{\overline{z}} f / \partial_z f = \mu$ a.e.
- Deduce that
  - f is conformal on Ω
  - f is not the restriction of a quasiconformal mapping of the whole sphere

An important lemma

Lemma

If Ω is quasiconformally rigid, then $\partial\Omega$ has zero area.

Assume $\partial\Omega$ has positive area.

- Construct a measurable function $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ such that
  - $\mu = 0$ on Ω, $\|\mu\|_{L^\infty(\overline{\Omega})} = 1$
  - μ satisfies the (strong) David condition :
    $m(\{z \in \widehat{\mathbb{C}} : |\mu(z)| > 1 - \epsilon\}) \leq Me^{-\beta e^{\frac{\alpha}{\epsilon}}} \qquad (\epsilon > 0).$
  - μ is **invariant** with respect to the Schottky group of Ω.
- Use David's theorem to obtain a homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ belonging to $W^{1,1}_{loc}$ with $\mu_f := \partial_{\overline{z}} f / \partial_z f = \mu$ a.e.
- Deduce that
  - f is conformal on Ω
  - f is not the restriction of a quasiconformal mapping of the whole sphere
  - $f(\Omega)$ is a circle domain.

Quasiconformally rigid implies conformally rigid

Quasiconformally rigid implies conformally rigid

Assume that Ω is quasiconformally rigid, and let f be a conformal map of Ω onto another circle domain.

Quasiconformally rigid implies conformally rigid

Assume that Ω is quasiconformally rigid, and let f be a conformal map of Ω onto another circle domain.

- Then f is the restriction of a quasiconformal mapping $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Quasiconformally rigid implies conformally rigid

Assume that Ω is quasiconformally rigid, and let f be a conformal map of Ω onto another circle domain.

- Then f is the restriction of a quasiconformal mapping $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.
- Use the fact that $\partial\Omega$ has zero area and results of Sullivan on Kleinian groups to deduce that g is a Möbius transformation.

THANK YOU!