Lower codimension-1 mass bounds in metric spaces

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- Classical Euclidean Isoperimetric inequality: If $E \subset \mathbb{R}^n$ is closed, then
$$\text{Area}(\partial E) \geq c_n \text{Vol}(E)^{\frac{n-1}{n}}.$$
- Metric space generalizations of closely-related filling inequalities (Almgren, Federer–Fleming, Gromov, Wenger). Mainly applicable to rectifiable spaces/objects.
- Analogs in many other contexts (Riemannian manifolds, sub-Riemannian manifolds, graph theory). Common theme is to bound Area(∂E) below by some function of Vol(E).

- Motivating question: Are there analogous statements for possibly "fractal" metric spaces, eg. for spaces quasisymmetrically equivalent to $\mathbb{R}^n$ or $\mathbb{S}^n$?
- Short answer: Yes, if one replaces Vol(E) by inradii of E and its complement.

Some definitions

Let (Z, d) be a metric space.

- For $E \subset Z$ Borel, define its inradius to be

$$\text{in-rad}(E) = \sup\{r \geq 0 : B(x,r) \subset E \text{ for some } x \in Z\}.$$

- For $S \subset Z$ closed, define its separation radius to be

$$\text{sep-rad}(S) = \sup\left\{\min(\text{in-rad}(U), \text{in-rad}(V))\right\}.$$

  taken over U, V distinct connected components of $Z \backslash S$.
- Soft relationship: if (M, d) is an n-dimensional metric manifold and $S \subset M$ is closed, then $\text{sep-rad}(S) > 0 \implies \mathcal{H}_{n-1}(S) > 0$.
- Z is called D-doubling if every ball $B(x,r)$ can be covered by $\leq D$ balls of radius $r/2$.
- Z is called L-linearly locally contractible if every ball $B(x,r)$ with $0 < r \leq \text{diam}(Z)/L$ can be contracted to a point inside of $B(x, Lr)$.

Theorem (K. '16)

Let (M, d) be a closed, connected, metric manifold of dimension $n \geq 2$ that is D-doubling and L-linearly locally contractible. For any closed set $S \subset M$,

$$\mathcal{H}_{n-1}(S) \geq c \cdot \text{sep-rad}(S)^{n-1},$$

where $c > 0$ depends only on n, D, and L.

Some consequences:

- "Quantitative topological" isoperimetric inequality: For (M, d) as above, if $E \subset M$ is Borel then

$$\mathcal{H}_{n-1}(\partial E) \geq c \cdot \min(\text{in-rad}(E), \text{in-rad}(M \backslash E))^{n-1}.$$

- Lower volume bounds for balls: For (M, d) as above,

$$\mathcal{H}_n(B(x, r)) \geq c \cdot r^n$$

for all $x \in M$ and $0 < r \leq \text{diam}(M)$.

Outline of Proof

Let (M, d) be a closed, connected, metric manifold of dimension $n \geq 2$ that is D-doubling and L-linearly locally contractible. Fix $S \subset M$ closed.

1. Not difficult to show that

   $$\text{sep-rad}(S) \approx \sup \{\text{dist}(S, \{x, y\}) : S \text{ separates } x \text{ and } y\}.$$

   Scaling M, assume that $\text{dist}(S, \{x, y\}) = 1$. Show that $\mathcal{H}_{n-1}(S) \gtrsim 1$.
2. Suppose $\mathcal{H}_{n-1}(S) \ll 1$. Approximate M by a simplicial complex $\mathcal{N} = \text{Ner}(\mathcal{U})$ using a well-chosen cover $\mathcal{U}$. There is a natural Lipschitz map $f \colon M \to \mathcal{N}$.
3. $\mathcal{H}_{n-1}(f(S))$ is still very small. Find a "projection" $p \colon \mathcal{N} \to \mathcal{N}$ with $p(f(S))$ in the $(n-2)$-skeleton of $\mathcal{N}$ (Federer–Fleming argument).
4. There is $g \colon \mathcal{N} \to M$, an "approximate inverse" to f, such that $g \circ p \circ f$ is very close to id_M.

The map $g \circ p \circ f$ only wiggles S a little bit, so the image should still separate x and y. However, $g \circ p \circ f|_S$ factors through an $(n-2)$-dimensional complex.

(M, d) as before, in particular D-doubling, L-linearly locally contractible, n-manifold. Note that $\operatorname{diam}(M) \geq 1$.

- For $\epsilon \ll 1$, take open cover $\mathcal{U} = \{U_i\}_{i=1}^{\ell}$ of M using balls of radius $\approx \epsilon$, with Lebesgue number $\approx \epsilon$, and multiplicity $\leq D$. Form the simplicial complex

$$\operatorname{Ner}(\mathcal{U}) = \bigcup \{\operatorname{conv}(e_{i_1}, \dots, e_{i_m}) : U_{i_1} \cap \dots \cap U_{i_m} \neq \emptyset\} \subset \mathbb{R}^{\ell}.$$

- Take Lipschitz partition of unity $\{f_i\}$ subordinate to $\mathcal{U}$ to obtain $f \colon M \to \operatorname{Ner}(\mathcal{U})$ via

$$f(x) = \sum_{i=1}^{\ell} f_i(x) e_i.$$

  Lipschitz constant is $\lesssim 1/\epsilon$.

- Define approximate inverse $g \colon \operatorname{Ner}(\mathcal{U}) \to M$ by $g(e_i) \in U_i$, then induction on skeleta to fill it in, using linear local contractibility. Note that $\operatorname{diam}(g(\sigma)) \lesssim \epsilon$ for all simplices $\sigma \subset \operatorname{Ner}(\mathcal{U})$. In particular,

$$d(g \circ f(x), x) \lesssim \epsilon \quad \text{for all } x \in M.$$

Federer–Fleming projection

Consider $f(S) \subset \text{Ner}(\mathcal{U})$, which has $\mathcal{H}_{n-1}(f(S)) \lesssim \epsilon^{-(n-1)}\mathcal{H}_{n-1}(S) \ll 1$.

- Starting with top-dimensional simplices in $\text{Ner}(\mathcal{U})$, want to consecutively project $f(S)$ to simplex boundaries, down to the $(n-2)$-skeleton.
- Key lemma: Let $E \subset \Delta_m$ closed with $\mathcal{H}_k(E) < \infty$ and $k < m$. There is $p \colon \Delta_m \to \Delta_m$, fixing $\partial\Delta_m$, with $p(E) \subset \partial\Delta_m$ and $\mathcal{H}_k(p(E)) \leq C_m\mathcal{H}_k(E)$.
- Idea: For $y \in \frac{1}{2}\Delta_m \backslash E$, radial projection p_y away from y has Lipschitz constant $\lesssim \text{dist}(y, K)^{-1}$ on each compact set K. This implies

$$\mathcal{H}_k(p_y(E)) \lesssim \int_E |x-y|^{-k} d\mathcal{H}_k(x)$$

  Integrating over $y \in \frac{1}{2}\Delta_m \backslash E$, we find y with $\mathcal{H}_k(p_y(E)) \lesssim \mathcal{H}_k(E)$.
- We know $\text{Ner}(\mathcal{U})$ has dimension $\leq D$. Consecutively project $f(S)$ to $(D-1)$-skeleton, $(D-2)$-skeleton, ..., $(n-1)$-skeleton. Increases $\mathcal{H}_{n-1}$-measure by a uniform multiplicative factor each time.
- As long as initial measure is small enough, after projections it is still $\ll 1$. The image can't fill an $(n-1)$-simplex, so radially project to $(n-2)$-skeleton.

Consider $g \circ p \circ f\colon M \to M$, which has $d(g \circ p \circ f(x), x) \lesssim \epsilon$ for all $x \in M$.

- S. Semmes: under the given conditions on M, there is $\delta > 0$ such that whenever $h\colon M \to M$ has $d(h(x), x) < \delta$ for all x, there is homotopy between h and id_M moving points by distance $\leq 1/4$.
- Taking ϵ small enough, depending only on the data, this applies to $h = g \circ p \circ f$. In particular, $h|_S$ is homotopic to $\iota\colon S \hookrightarrow M$ through maps that do not meet $\{x, y\}$.
- On cohomology:
  1. The induced homomorphism
     $$h^*\colon \check{H}^{n-1}(h(S)) \to \check{H}^{n-1}(S)$$
     is non-trivial (in particular, both groups are non-trivial).
  2. On the other hand,
     $$\begin{array}{ccc} \check{H}^{n-1}(h(S)) & \xrightarrow{h^*} & \check{H}^{n-1}(S) \\ g^* \searrow & & \nearrow (p \circ f)^* \\ & \check{H}^{n-1}(\mathrm{Ner}(\mathcal{U})^{(n-2)}) = 0 & \end{array}$$
- Contradiction! So $\mathcal{H}_{n-1}(S) \gtrsim \epsilon^{n-1} \gtrsim 1$.