Fractal Convolution Inequalities and Geometric Applications

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$$T_\lambda f(x) = \lambda * (f\mu),$$

where

- μ and ν are compactly supported Borel measures satisfying

$$\mu(B(x,r)) \leq Cr^{s_\mu} \text{ and } \nu(B(x,r)) \leq Cr^{s_\nu}.$$

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- The key to Falconer's result can be expressed as an estimate on the $L^1(\mu)$ norm of the operator

$$T_\sigma f(x) = \sigma * (f\mu)(x).$$

Finite point configurations

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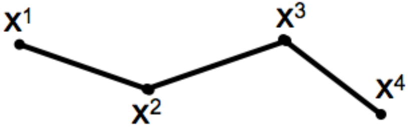

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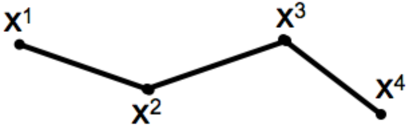

Definition

A k-chain in $E \subset \mathbb{R}^d$ with gaps $\{t_i\}_{i=1}^k$ is a sequence

$$\{x^1, x^2, \dots, x^{k+1} : x^j \in E;\ |x^{i+1} - x^i| = t_i;\ 1 \le i \le k\}.$$

We say that the chain is *non-degenerate* if all the x^js are distinct.

Theorem (Bennett, Iosevich, K.T.)

Suppose that the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$. Then for any $k \geq 1$, there exists an open interval I, such that for any $t \in I$ there exists a non-degenerate k-chain in E with gap lengths equal to t.

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- The idea behind the proof is to construct a measure on all $k-chains$.
- We bound

$$C_k^{\epsilon}(\mu) = \int \left(\int \cdots \int \prod_{i=1}^{k} \sigma_t^{\epsilon}(x^{i+1} - x^i) d\mu(x^i) \right) d\mu(x^{k+1})$$

from above (for all values of $t > 0$) and below (in the case when t is in a suitable interval).

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Theorem (Iosevich, Krause, Sawyer, K.T., Uriarte-Tuero)

Suppose that μ, ν are compactly supported Borel measures on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^{s_\mu}, \nu(B(x,r)) \leq Cr^{s_\nu}$, respectively, with $s_\mu + s_\nu > d + 1$. Let

$$T_t f = \sigma_t * (f\mu).$$

Then

$$||T_t f||_{L^2(\nu)} \leq C||f||_{L^2(\mu)}.$$

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- We also prove a maximal version of this theorem. This can be viewed as a fractal variant of Stein's spherical maximal theorem.

Circle Problem

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- Theorem: If $dim(U) > 1$, then E has positive Lebesgue measure.
- This problem can be studied a number of ways.
- In particular, it follows from the inequality: $||\sigma_t * (f\mu)||_{L^2(\nu)} \leq C||f||_{L^2(\mu)}$ where μ and ν are as above.

Pinned distance set

- The $L^2(\mu)$ mapping properties of these fractal convolution operators can also be used to recover and extend the pinned distance set result due to Peres and Schlag.

THANK YOU