On the Geometry of Rectifiable Sets with Carleson and Poincaré-type Conditions

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- In 1960, Reifenberg proved that if a set is well approximated by n-planes, then it is a homeomorphic (more precisely bi-Hölder) image of an n-plane.

- In 2012, David and Toro proved that if the oscillations of these approximating n-planes are controlled, then the set is a bi-Lipschitz image of an n-plane.

- Smooth surfaces of co-dimension 1 whose oscillation of the unit normal is small are called CASSC. They were introduced by Semmes in 1991.

- It is still an open question if CASSC admit a bi-Lipschitz parametrization.

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The setting:

- We consider M, an n-rectifiable subset of $\mathbb{R}^{n+1}$.
- We ask that M be Ahlfors regular:

Definition

An $\mathcal{H}^n$-measurable set M is called Ahlfors regular if it is closed and there exists a constant $C \geq 1$ such that

$$C^{-1} r^n \leq \mathcal{H}^n(B_r(x) \cap M) \leq C\, r^n$$

for all $x \in M$ and $r > 0$.

- Let $\mu = \mathcal{H}^n \llcorner M$, the n-dimensional Hausdorff measure restricted to M.

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Carleson Condition on the unit normals to M

We consider the following Carleson-type condition on the oscillation of the unit normal ν to our rectifiable set M:

For all $x \in M$, we have

$$\int_0^1 \left(\fint_{B_r(x)} |\nu(y) - \nu_{x,r}|^2 \, d\mu \right) \frac{dr}{r} < \epsilon \,, \tag{1}$$

where $\nu_{x,r} = \fint_{B_r(x)} \nu(y) \, d\mu(y)$ is the average of the unit normal ν on $B_r(x)$,and where ϵ is a small number to be determined later.

Poincaré Inequality on M

We consider the following Poincaré-type inequality on our rectifiable set M:

For all $x \in M$, $r > 0$, and for *any* locally Lipschitz function f on $\mathbb{R}^{n+1}$, we have

$$\fint_{B_r(x)} |f(y) - f_{x,r}| \, d\mu(y) \leq c_P \, r \left(\fint_{B_{2r}(x)} |\nabla^M f(y)|^2 \, d\mu(y) \right)^{\frac{1}{2}}, \tag{2}$$

where c_P denotes the Poincaré constant, which is a constant depending only on n, $f_{x,r} = \fint_{B_r(x)} f(y) \, d\mu(y)$ is the average of the function f on $B_r(x)$, and $\nabla^M f(y) = p_{T_y M}(\nabla f(y))$ denotes the tangential derivative of f.

Bi-Lipschitz Parametrization of M

Theorem (M., 2015)

Let $M \subset B_1(0)$ be an n-Ahlfors regular rectifiable set containing the origin. Assume that M satisfies the Poincaré inequality (2) and the unit normal ν to M satisfies the Carleson-type condition (1) with an $\epsilon > 0$ (small enough) that depends only on n.

Then, there exists a bijective K-bi-Lipschitz map $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ where the bi-Lipschitz constant K depends only on n, and an n-dimensional plane Σ, such that

$$g(\Sigma) \text{ is a } \epsilon\text{-Reifenberg flat set, and}$$

$$M \cap B_{\frac{1}{2}}(0) \subset g(\Sigma).$$

Ideas of the proof

- Recall Carleson-type Condition

$$\int_0^1 \left(\fint_{B_r(x)} |\nu(y) - \nu_{x,r}|^2 \, d\mu \right) \frac{dr}{r} < \epsilon$$

- Define $P_{x,r}$ to have unit normal $\nu_{x,r}$.
- Poincare Inequality $\implies$ $P_{x,r}$ good approximating n-plane.
- Carleson Condition $\implies$ oscillations of $P_{x,r}$ is controlled.

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Quasiconvexity of M

Notice that the containment in the above result is because M might be full of holes. It turns out that these holes cannot be too big.

Definition

A space X in $\mathbb{R}^{n+1}$ is quasiconvex if there exists a constant $\kappa \geq 1$ such that for any two points x and y in X, there exists a rectifiable curve γ in X, joining x and y, such that length$(\gamma) \leq \kappa |x - y|$.

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Further Results and Current Projects

- The above results also hold in higher co-dimensions when considering a Carleson-type condition on the oscillation of the tangent planes of the rectifiable set M.
- There are examples of non-smooth surfaces that satisfy the Poincaré-type inequality.
- It seems that the Poincaré inequality does not get rid of the holes in M.

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Thank You!