The Analyst's Traveling Salesman theorem in graph inverse limits

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- Define

$$\beta_E(Q) = \frac{1}{\text{diam}(Q)} \inf_L \sup_{x \in E \cap Q} \text{dist}(x, L)$$

where the infimum is taken over all lines L in the plane

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Theorem (Jones '90)

(a) (Upper Bound) *If* $\Gamma \subset \mathbb{R}^2$ *is connected, then*

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(b) (Construction) *If* $E \subset \mathbb{R}^2$ *is any set, then* E *is contained in a connected set* Γ *satisfying*

$$\mathcal{H}^1(\Gamma) \lesssim \text{diam}(E) + \sum_{\text{dyadic cubes } Q} \beta_E(3Q)^2 \text{diam}(Q).$$

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- Ferrari-Franchi-Pajot '07: Generalization to the Heisenberg group, with β-numbers measured with respect to **horizontal lines**.
- Li-Schul '14, '15: Improved generalization to Heisenberg group, showing that the relevant exponent is 4, not 2.

Lessons from the Heisenberg group story

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- the correct exponent(s).

Definition of the spaces (Cheeger-Kleiner)

Our space X will be an inverse limit of connected simplicial metric graphs:

$$X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} \dots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} \dots$$

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3. If X_i' denotes the graph obtained by subdividing each edge of X_i into m edges of length $m^{-(i+1)}$, then π_i induces a map $\pi_i : (X_{i+1}, d_{i+1}) \to (X_i', d_i)$ which is open, simplicial, and an isometry on every edge.

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4. For every $x_i \in X_i'$, the inverse image $\pi_i^{-1}(x_i) \subset X_{i+1}$ has d_{i+1}-diameter at most ηm^{-i}.

Example 1: The Lang-Plaut example

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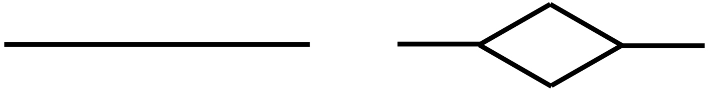

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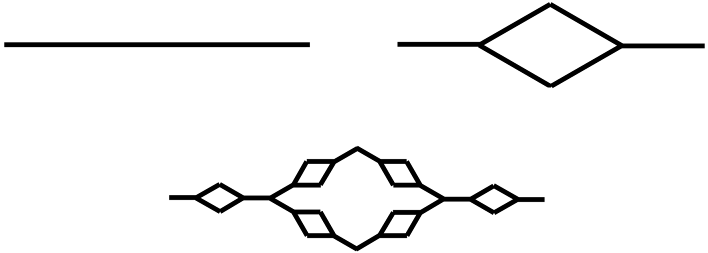

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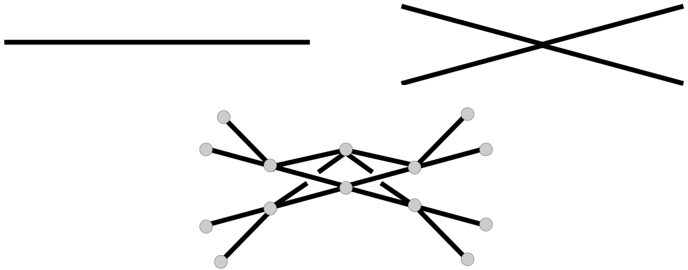

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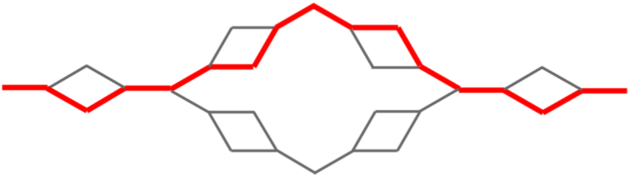

β-numbers in these spaces

Let E be a subset of X, and let B be a ball in X.

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We define

$$\beta_E(B) = \frac{1}{\mathrm{diam}(B)} \inf_L \sup_{x \in E \cap B} \mathrm{dist}(x, L)$$

where the supremum is taken over all **monotone geodesics** L in X.

The upper bound

Fix a space X as above, and an appropriate m-adic system $\mathcal{G}$ of balls in X.

Theorem (Upper bound)

For every $p > 1$, there is a constant C_p such that, if $\Gamma \subset X$ is connected, then

$$\sum_{B \in \mathcal{G}} \beta_\Gamma(B)^p \operatorname{diam} B \leq C_p \mathcal{H}^1(\Gamma).$$

The constant C_p depends only on p and the constants associated to the construction of X.

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The exponent is sharp: there is a counterexample for $p = 1$.

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Theorem (Construction)

There are constants $C > 1$ and $\epsilon > 0$, depending only on the data of X, with the following property: Let $E \subset X$ be compact. Then there is a compact connected set $\Gamma \subset X$ containing E such that

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Remark: This implies that

$$\mathcal{H}^1(\Gamma) \leq C_p \left(\operatorname{diam}(E) + \sum_{B \in \mathcal{G}} \beta_E(B)^p \operatorname{diam}(B) \right),$$

where C_p depends only on $p > 0$ and the data of X.

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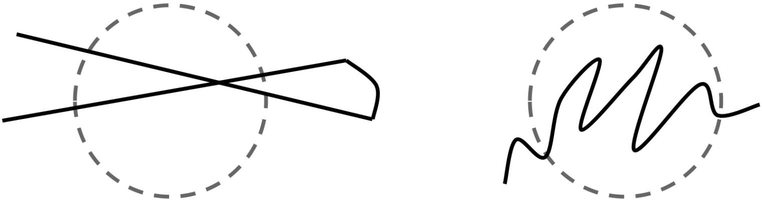

(a) A $\mathcal{G}_1$ ball.

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- The idea here is that, due to the discrete approximation of the space, if the parametrization γ passes through a "non-flat ball", $\pi_0 \circ \gamma$ must backtrack, and there is a quantitative bound on how much a real-valued Lipschitz function can backtrack.

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- At locations in Γ_{i-1} with large β, it is clear what to do: just take all possible lifts.
- At locations with small β, one must be careful to take an essentially optimal lift and maintain its connectedness to the rest of the curve. This is where the majority of the technical problems come in.