A Measure Zero Universal Differentiability Set in the Heisenberg Group

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Rademacher's Theorem

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Every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable Lebesgue almost everywhere.*

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Question: Suppose $N \subset \mathbb{R}^n$ is Lebesgue null. Does there exist a Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}^m$ which is differentiable at no point of N?

Converse to Rademacher's Theorem?

Theorem ($n = 2$: Alberti, Csörnyei, Preiss. $n > 2$: ACP+ C., Jones)

If $N \subset \mathbb{R}^n$ is Lebesgue null then there is a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ which is differentiable at no point of N.

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Theorem (Preiss)

If $n > 1$, there exists a Lebesgue null set $N \subset \mathbb{R}^n$ such that every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at some point of N.

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Theorem (Doré-Maleva, Dymond-Maleva)

The ***universal differentiability set*** *N above can be made compact and of Hausdorff dimension, or even upper Minkowski dimension, equal to one.*

Maximality of Directional Derivatives

Let E be a Banach space.

Theorem (Fitzpatrick)

Suppose $f\colon E \to \mathbb{R}$ *is Lipschitz and* $f'(x,e) = \mathrm{Lip}(f)$ *for some* $x \in E$ *and* $e \in E$ *with* $\|e\| = 1$*. If the norm of* E *is Fréchet differentiable at* e *with derivative* e^**, then* f *is Fréchet differentiable at* x *and* $f'(x) = \mathrm{Lip}(f)e^*$*.*

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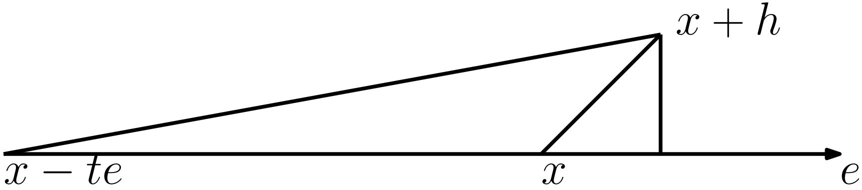
Suppose f is not differentiable at x - find $\varepsilon > 0$ and small h such that:

$$f(x+h) - f(x) > \mathrm{Lip}(f)e^*(h) + \varepsilon\|h\|.$$



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Suppose $f \colon E \to \mathbb{R}$ is Lipschitz and $(x_0, e_0) \in D^f$. Let M denote the set of all pairs $(x, e) \in D^f$ such that $f'(x, e) \geq f'(x_0, e_0)$ and

$$|(f(x + te_0) - f(x)) - (f(x_0 + te_0) - f(x_0))| \leq 6|t|\sqrt{(f'(x, e) - f'(x_0, e_0))\mathrm{Lip}(f)}$$

for every $t \in \mathbb{R}$.

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for every $t \in \mathbb{R}$. If the norm is Fréchet differentiable at e_0 and

$$\lim_{\delta \downarrow 0} \sup\{f'(x,e) \colon (x,e) \in M \text{ and } \|x - x_0\| \leq \delta\} \leq f'(x_0,e_0),$$

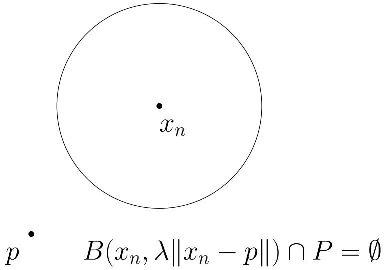
then f is Fréchet differentiable at x_0.

Porous Sets

Definition

Let (X, d) be a metric space. A set $P \subset X$ is **porous** if there is $\lambda > 0$ such that for every $p \in P$: there is a sequence $x_n \in X$ with $x_n \to p$ and $B(x_n, \lambda \|x_n - p\|) \cap P = \varnothing$.

A set is σ-**porous** if it is a countable union of porous sets.



Applications of Porosity

Lemma (Lindenstrauss, Preiss)

Suppose $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is Lipschitz. Then the following implication holds outside a σ-porous set:
If f is differentiable at x in the direction of an $(n-1)$-dimensional plane T then f is ***regularly differentiable*** *at x in the direction of T.*

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There exists a Lebesgue null set $N \subset \mathbb{R}^n$ such that every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is differentiable at a point of N.

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Theorem (Preiss, Tiser, Zajicek)

Suppose $P \subset \mathbb{R}^n$ *is* σ*-porous. Then there is a Lipschitz function* $f : \mathbb{R}^n \to \mathbb{R}$ *which is differentiable at no point of* P*. Hence a universal differentiability set in* $\mathbb{R}^n$ *cannot be* σ*-porous.*

Heisenberg Group

Definition

The **Heisenberg group** $\mathbb{H}^n$ is the set $\mathbb{R}^{2n+1}$ equipped with the group law:

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2(\langle x, y' \rangle - \langle y, x' \rangle)).$$

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Left-invariant **horizontal vector fields** on $\mathbb{H}^n$ are defined by:

$$X_i(x, y, t) = \partial_{x_i} + 2y_i \partial_t, \quad Y_i(x, y, t) = \partial_{y_i} - 2x_i \partial_t, \quad 1 \leq i \leq n.$$

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- **Haar measure** on $\mathbb{H}^n$ is $\mathcal{L}^{2n+1}$.
- **Dilations** are defined by $\delta_r(x, y, t) = (rx, ry, r^2 t)$.

Horizontal Curves and Carnot-Carathéodory Distance

Definition

An absolutely continuous curve $\gamma\colon [a, b] \to \mathbb{H}^n$ is **horizontal** if there exists $h\colon [a, b] \to \mathbb{R}^{2n}$ such that for almost every t:

$$\gamma'(t) = \sum_{i=1}^{n} h_i(t) X_i(\gamma(t)) + h_{i+n}(t) Y_i(\gamma(t)).$$

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- d_{cc} is not Lipschitz equivalent to the Euclidean distance.
- Horizontal curves in $\mathbb{H}^n$ are lifts of curves in $\mathbb{R}^{2n}$.

Pansu Differentiability

Definition

A function $L\colon \mathbb{H}^n \to \mathbb{R}$ is called $\mathbb{H}$-linear if $L(xy) = L(x) + L(y)$ and $L(\delta_r(x)) = rL(x)$ for all $x, y \in \mathbb{H}^n$ and $r > 0$.

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Theorem (Pansu)

Every Lipschitz function $f\colon \mathbb{H}^n \to \mathbb{R}$ (or even between general Carnot groups) is Pansu differentiable Lebesgue almost everywhere.

A Lebesgue Null Universal Differentiability Set in $\mathbb{H}^n$

Theorem (Pinamonti, S.)

There is a Lebesgue null 'universal differentiability set' $N \subset \mathbb{H}^n$ such that every Lipschitz function $f : \mathbb{H}^n \to \mathbb{R}$ is Pansu differentiable at a point of N.

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1. Fix a Lebesgue null G_δ set N containing all horizontal lines joining pairs of points in $\mathbb{Q}^{2n+1}$.

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2. Find an 'almost maximal' directional derivative $Ef(x)$, where we consider $x \in N$ and horizontal vector fields E of unit length.
3. Show that if $x \in N$ and $Ef(x)$ is 'almost maximal' then f is Pansu differentiable at x.

Directional Derivatives in $\mathbb{H}^n$

Let $V = \mathrm{Span}\{X_i, Y_i \colon 1 \leq i \leq n\}$.
Let ω be the inner product norm on V for which X_i, Y_i are orthonormal.

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Definition

Let $f : \mathbb{H}^n \to \mathbb{R}$ be Lipschitz and $E \in V$. Define $Ef(x) := (f \circ \gamma)'(t)$ whenever it exists, where γ is any Lipschitz horizontal curve with $\gamma(t) = x$ and $\gamma'(t) = E(x)$.

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Lemma

Let $f : \mathbb{H}^n \to \mathbb{R}$ be Lipschitz. Then:

$$\mathrm{Lip}_{\mathbb{H}}(f) = \sup\{|Ef(x)| : x \in \mathbb{H}^n, E \in V, \omega(E) = 1, Ef(x) \text{ exists }\}.$$

Maximality implies Differentiability

Lemma

Fix $u_1, u_2 \in \mathbb{R}^n$ *not both zero and let* $u = (u_1, u_2, 0) \in \mathbb{H}^n$*. Then:*

1. $d_{cc}(uz, 0) \geq d_{cc}(u, 0) + \langle z, u/d_{cc}(u, 0)\rangle$ *for any* $z \in \mathbb{H}^n$*,*

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   That is, the Pansu derivative of $d_{cc}(\cdot, 0)$ *at* u *is* $x \mapsto \langle x, u/d_{cc}(u, 0)\rangle$.

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Theorem

Let $f : \mathbb{H}^n \to \mathbb{R}$ *be Lipschitz,* $x \in \mathbb{H}^n$ *and* $E \in V$ *with* $\omega(E) = 1$. *Suppose* $Ef(x)$ *exists and* $Ef(x) = \mathrm{Lip}_{\mathbb{H}}(f)$. *Then* f *is Pansu differentiable at* x *with derivative* $x \mapsto \mathrm{Lip}_{\mathbb{H}}(f)\langle x, E(0)\rangle$.

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for every $t \in (-1, 1)$.

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for every $t \in (-1, 1)$. If

$$\lim_{\delta \downarrow 0} \sup\{Ef(x) : (x, E) \in M \text{ and } d_{cc}(x, x_0) \leq \delta\} \leq E_0 f(x_0),$$

then f is Pansu differentiable at x_0 with derivative $x \mapsto E_0 f(x_0)\langle x, E_0(0)\rangle$.

Carnot Groups

Definition

A Carnot group $\mathbb{G}$ is a simply connected Lie group whose Lie algebra $\mathcal{G}$ admits a stratification:

$$\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

with

$$[V_1, V_i] = V_{i+1} \text{ if } 1 \leq i \leq s-1 \text{ and } [V_1, V_s] = 0.$$

Example

Euclidean spaces are Carnot groups of step $s = 1$.
The Heisenberg group is a Carnot group of step $s = 2$.

Carnot groups admit structures like those on the Heisenberg group: translations, dilations, Haar measure, horizontal curves, Carnot-Carathéodory distance, Pansu's theorem...

Porosity and Differentiability in Carnot Groups

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Suppose $f : \mathbb{G} \to \mathbb{R}$ is Lipschitz. Then there is a σ-porous set $A \subset \mathbb{G}$ such that for every $x \notin A$:

1. *if $E_1 f(x)$ and $E_2 f(x)$ exist for some $E_1, E_2 \in V_1$ then $(a_1 E_1 + a_2 E_2) f(x)$ exists and is equal to $a_1 E_1 f(x) + a_2 E_2 f(x)$,*
2. *x is a regular point of f.*

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Suppose $f : \mathbb{G} \to \mathbb{R}$ is Lipschitz and $x \in \mathbb{G}$. Assume (1) and (2) above hold and $X_1 f(x), \ldots, X_m f(x)$ exist for a basis $X_1, \ldots, X_m$ of V_1. Then f is Pansu differentiable at x.

Porosity and Differentiability in Carnot Groups

Theorem (Pinamonti, S.)

Suppose $f : \mathbb{G} \to \mathbb{R}$ is Lipschitz. Then there is a σ-porous set $A \subset \mathbb{G}$ such that for every $x \notin A$:

1. *if $E_1 f(x)$ and $E_2 f(x)$ exist for some $E_1, E_2 \in V_1$ then $(a_1 E_1 + a_2 E_2) f(x)$ exists and is equal to $a_1 E_1 f(x) + a_2 E_2 f(x)$,*
2. *x is a regular point of f.*

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Corollary (Pansu's theorem for Euclidean targets)

Every Lipschitz map $f : \mathbb{G} \to \mathbb{R}^n$ is Pansu differentiable almost everywhere.

Porosity and Differentiability in Carnot Groups

Theorem (Pinamonti, S.: work in progress)

Let $P \subset \mathbb{G}$ be σ-porous. Then there is a Lipschitz function $f : \mathbb{G} \to \mathbb{R}$ which is Pansu differentiable at no point of P.

Porosity and Differentiability in Carnot Groups

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Universal differentiability sets in Carnot groups cannot be σ-porous.

Questions:

- Do measure zero universal differentiability sets exist in all Carnot groups?
- Can one adapt techniques of Doré, Dymond and Maleva to construct, in Carnot groups, compact universal differentiability sets of small dimension?

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- Measure zero 'universal differentiability sets' in $\mathbb{H}^n$ contain points of Pansu differentiability for real-valued Lipschitz functions.
- Connections between porosity and differentiability in the linear setting generalise to general Carnot groups.

Thank you for listening!